DIRAC OPERATORS IN TENSOR CATEGORIES AND THE MOTIVE OF QUATERNIONIC MODULAR FORMS

MARC MASDEU AND MARCO ADAMO SEVESO

Abstract. We define a motive whose realizations afford modular forms (of arbitrary weight) on an indefinite division quaternion algebra. This generalizes work of Iovita–Spiess to odd weights in the spirit of Jordan–Livné. It also generalizes a construction of Scholl to indefinite division quaternion algebras, and provides the first motivic construction of new-subspaces of modular forms.

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1. Introduction

The paper [Sc] offers the construction of a motive whose realizations afford modular forms of even or odd weight on the indefinite split quaternion algebra over $\mathbb{Q}$. In [IS, §10] the authors construct a motive of even weight modular forms on a quaternion division algebra (see also [Wo]). Based on ideas of Jordan and Livné (see [JL]), this motive is constructed as the kernel of an appropriate Laplace operator. More precisely, let $h(A)$ be the motive of an abelian scheme A of relative dimension d over a smooth base scheme S (see [DM] and [Ku]). It decomposes as the direct sum

$$h\left(A\right)=h^{0}\left(A\right)\oplus h^{1}\left(A\right)\oplus ...\oplus h^{g}\left(A\right)$$

where $g=2d$ and there are canonical identifications

$$h^{i}\left(A\right)=\vee^{i}h^{1}\left(A\right),\ h^{i}\left(A\right)\simeq h^{2d-i}\left(A\right)^{\vee}\left(-d\right)\ \text{ and }h^{2d}\left(A\right)\simeq\mathbb{I}\left(-d\right),$$

where $\vee^{\cdot}V$ denotes the symmetric algebra of the object V. It follows that the multiplication morphisms

$$\varphi_{i,2d-i}:\vee^{i}h^{1}\left(A\right)\otimes\vee^{2d-i}h^{1}\left(A\right)\rightarrow\mathbb{I}\left(-d\right)$$

are perfect. In particular, taking $i=d$, one gets an associated Laplace operator[1]

$$\Delta^{n}:\mathrm{Sym}^{n}\left(\vee^{d}h^{1}\left(A\right)\right)\rightarrow\mathrm{Sym}^{n-2}\left(\vee^{d}h^{1}\left(A\right)\right)\left(-d\right),\ n\geq2$$

and it is possible to show that the kernel exists. The following remark has been employed in [IS, §10]. When A is an abelian scheme of dimension $d=2$ with multiplication by the quaternion algebra B, we have that $B\otimes B$ acts on $\vee^{2}h^{1}\left(A\right)$ and there is a canonical direct sum decomposition

$$\vee^{2}h^{1}\left(A\right)=\left(\vee^{2}h^{1}\left(A\right)\right)_{+}\oplus\left(\vee^{2}h^{1}\left(A\right)\right)_{-}$$

[1]For a symmetric or alternating power M we will write $\mathrm{Sym}^{n}\left(M\right)$ and $\mathrm{Alt}^{n}\left(M\right)$ when considering its symmetric or alternating powers once again.

is such a way that $B^{\times} \subset B \otimes B$ (diagonally) acts via the reduced norm on $\left(\vee^2 h^1(A)\right)_-$. Furthermore, since the idempotents giving rise to this decomposition are self-adjoint with respect to $\varphi_{2,2}$, it follows that the induced pairing

$$\left(\vee^2 h^1(A)\right)_- \otimes \left(\vee^2 h^1(A)\right)_- \hookrightarrow \vee^2 h^1(A) \otimes \vee^2 h^1(A) \to \mathbb{I}(-2)$$

is still non-degenerate and the kernel of the induced Laplace operator

$$\Delta_-^n : \mathrm{Sym}^n\left(\left(\vee^2 h^1(A)\right)_-\right) \to \mathrm{Sym}^{n-2}\left(\left(\vee^2 h^1(A)\right)_-\right)(-2),\ n \geq 2$$

exists. When A is taken to be the universal abelian surface, setting

$$M_{2n} := \ker(\Delta^n)$$

gives a motive whose realizations gives incarnations of weight $k = 2n+2$ modular forms.

The aim of this paper is to define a motive whose realizations afford modular forms (of arbitrary weight) on an indefinite division quaternion algebra. The idea of the construction, once again, is due to Jordan and Livné. However some remarks are in order. First, it is worth noting that although the realizations of the motive constructed in this paper are abstractly isomorphic to the $D = \mathrm{disc}(B)$-new part of (two copies of) the realizations of the motive constructed in [Sc] via the Jacquet–Langlands correspondence, a "motivic Jacquet–Langlands correspondence" has not yet described that lifts this correspondence to the motivic setting. Therefore what we propose is the first construction –as a Chow motive– of D-new modular forms. Second, following their construction in this indefinite setting and working at the level of realizations gives the various incarnations of two copies of odd weight modular forms, rather than just one copy. It is not possible to canonically split them in a single copy: this is possible only including a splitting field for the quaternion algebra in the coefficients, but the resulting splitting depends on the choice of an identification of the base changed algebra with the split quaternion algebra. Indeed, we will construct a motive whose realizations afford two copies of odd weight modular forms. Finally, the idea of Jordan and Livné is to construct square roots of the Laplace operators after appropriately tensoring the source and the targets of Δ^n; however the definition of these Dirac operators ∂_{JL}^n such that $\partial_{JL}^{n-1} \circ \partial_{JL}^n = \Delta^n \otimes 1_?$ does not readily generalize to the setting of a rigid $\mathbb{Q}$-linear and pseudo-abelian ACU category like that of motives. To understand the linear algebra behind their construction, let us consider the category $\mathrm{Rep}(B^{\times})$ of algebraic $B^{\times}$-representations: let B (resp. B^{ι}) be the $B^{\times}$-representation whose underlying vector space is B on which $B^{\times}$ acts by left multiplication (resp. $b \cdot x = bxb^{\iota}$, where $b \mapsto b^{\iota}$ denotes the main involution) and set $B_0 := \ker(\mathrm{Tr}) \subset B^{\iota}$. Then the trace form $\langle x, y\rangle := \mathrm{Tr}(x^{\iota}y)$ induces $B^{\iota} \otimes B^{\iota} \to \mathbb{Q}(-2)$ and $B \otimes B \to \mathbb{Q}(-1)$ and the first is perfect when restricted to B_0 and gives Laplace operators

$$\Delta_-^n : \mathrm{Sym}^n(B_0) \to \mathrm{Sym}^{n-2}(B_0)(-2),\ n \geq 2, \tag{1}$$

while the second gives

$$B = B^{\vee}(-1). \tag{2}$$

We may realize $B_0 = \left(\wedge^2 B\right)_-$ and it follows that (1) may be regarded as

$$\Delta_-^n : \mathrm{Sym}^n\left(\left(\wedge^2 B\right)_-\right) \to \mathrm{Sym}^{n-2}\left(\left(\wedge^2 B\right)_-\right)(-2),\ n \geq 2.$$

Following ideas of [IS, §10], one can realize the various incarnation of modular forms as the image via an appropriate additive ACU tensor functor

$$\mathcal{L} : \mathrm{Rep}\left(B^{\times}\right) \to \mathcal{H},$$

where $\mathcal{H}$ is the category we are interested in, i.e. they may be for example variations of Hodge structures. Indeed, if R is the realization functor one shows that $R\left(\left(\vee^2 h^1(A)\right)_-\right) = \mathcal{L}\left(\left(\wedge^2 B\right)_-\right)$, from which it follows

$$R(M_{2n}) = \mathcal{L}\left(\ker\left(\Delta_-^n\right)\right)$$

and $\mathcal{L}(\ker(\Delta^n))$ computes weight $2n+2$ modular forms. If one is interested in *odd* weight modular forms, the Jordan and Livné Dirac operators to be considered would be of this form:

$$\partial_{JL}^n : \mathrm{Sym}^n(B_0) \otimes B \to \mathrm{Sym}^{n-1}(B_0) \otimes B(-1),\ n \geq 1. \tag{3}$$

However, as we have explained above, the Jordan and Livné definition of these operators as given in [JL] does not generalize to motives. It is a simple but key remark that one may replace ∂_{JL}^n with any ∂^n having the same source and target and then the kernels would be the same (see Lemma 7.6). Furthermore, since $B_0 = \left(\wedge^2 B\right)_-$, it follows from (2) that (3) with ∂^n_{JL} replaced by ∂^n may be regarded as

$$\partial^n : \mathrm{Sym}^n \left(\left(\wedge^2 B\right)_-\right) \otimes B \to \mathrm{Sym}^{n-1} \left(\left(\wedge^2 B\right)_-\right) \otimes B^\vee (-2), \, n \geq 1. \tag{4}$$

It is in this form that we will be able to define ∂^n and another $\overline{\partial}^{n-1}$ in such a way that the construction makes sense for rigid $\mathbb{Q}$-linear and pseudo-abelian ACU categories and prove the generalization of the equality $\overline{\partial}^{n-1} \circ \partial^n = \Delta^n_- \otimes 1_B$ in this setting. Then one shows that $\mathcal{L}(\partial^n)$ computes two copies of weight $k = 2n+3$ modular forms.

The abstract framework we work with in this paper is the following. Suppose that $\mathcal{C}$ is a rigid pseudo-abelian and $\mathbb{Q}$-linear ACU tensor category with identity object $\mathbb{I}$; if $X \in \mathcal{C}$ we write $r_X := \mathrm{rank}(X)$. We recall from [MS] that V has *alternating* (resp. *symmetric*) rank $g \in \mathbb{N}_{\geq 1}$ if $L := \wedge^g V$ (resp. $L := \vee^g V$) is invertible and if $\binom{r+i-g}{i}$ (resp. $\binom{r+g-1}{i}$) is invertible in $End(\mathbb{I})$ for every $0 \leq i \leq g$. Here, for an integer $k \geq 1$,

$$\binom{T}{k} := \frac{1}{k!} T(T-1) \ldots (T-k+1) \in \mathbb{Q}[T] \text{ and } \binom{T}{0} = 1.$$

Suppose first that V has alternating rank g. We will prove that, when $g = 2i$ and i is even (resp. odd), $L \simeq \mathbb{L}^{\otimes 2}$ for some invertible object and $r_{\wedge^i V} > 0$ (resp. $r_{\wedge^i V} < 0$) (see definition 3.6), then there is an operator

$$\partial^n_{i-1} : \mathrm{Sym}^n \left(\wedge^i V\right) \otimes \wedge^{i-1} V \to \mathrm{Sym}^{n-1} \left(\wedge^i V\right) \otimes V^\vee \otimes L, \, n \geq 1$$
$$(\text{resp. } \partial^n_{i-1} : \mathrm{Alt}^n \left(\wedge^i V\right) \otimes \wedge^{i-1} V \to \mathrm{Alt}^{n-1} \left(\wedge^i V\right) \otimes V^\vee \otimes L, \, n \geq 1)$$

such that $\ker\left(\partial^n_{i-1}\right)$ exists (see Theorems 4.4 and 4.3).

Suppose now that V has symmetric rank g. Then we prove that, when $g = 2i$, $L \simeq \mathbb{L}^{\otimes 2}$ for some invertible object and $r_{\vee^i V} > 0$, then there is an operator

$$\partial^n_{i-1} : \mathrm{Sym}^n \left(\vee^i V\right) \otimes \vee^{i-1} V \to \mathrm{Sym}^{n-1} \left(\vee^i V\right) \otimes V^\vee \otimes L, \, n \geq 1$$

such that $\ker\left(\partial^n_{i-1}\right)$ exists (see Theorem 5.3).

These operators are indeed square roots of the Laplace operators induced by the multiplication pairings in the involved alternating or symmetric algebras and the existence of these kernels follows from this fact and the existence of the kernels of the Laplace operators.

Some remarks are in order about the range of applicability of our results. First of all we note that, in general, the alternating or the symmetric rank may be not uniquely determined. Suppose, however, that we know that there is a field K such that $r \in K \subset End(\mathbb{I})$ admitting an embedding $\iota : K \hookrightarrow \mathbb{R}$. Then it follows from the formulas $\mathrm{rank}\left(\wedge^k V\right) = \binom{r}{k}$ and $\mathrm{rank}\left(\vee^k V\right) = \binom{r+k-1}{k}$ (see [AKh, 7.2.4 Proposition] or [De3, (7.1.2)]) that we have $r \in \{-1, g\}$ (resp. $r \in \{-g, 1\}$) when V has alternating (resp. symmetric) rank g. In particular, when $r > 0$ (resp. $r < 0$) with respect to the ordering induced by ι, we deduce that $r = g$ (resp. $r = -g$), so that g is a uniquely determined and V has alternating (resp. symmetric) rank $g = r$ (resp. $g = -r$)

We recall that V is Kimura positive (resp. negative) when $\wedge^{N+1} V = 0$ (resp. $\vee^{N+1} V = 0$) for $N \geq 0$ large enough. In this case, the formula $\mathrm{rank}\left(\wedge^k V\right) = \binom{r}{k}$ (resp. $\mathrm{rank}\left(\vee^k V\right) = \binom{r+k-1}{k}$) implies that $r \in \mathbb{Z}_{\geq 0}$ (resp. $r \in \mathbb{Z}_{\leq 0}$) and the smallest integer N such that $\wedge^{N+1} V = 0$ (resp. $\vee^{N+1} V = 0$) is r (resp. $-r$). Furthermore, it is known that $\wedge^r V$ (resp. $\vee^{-r} V$) is invertible in this case (see [Kh, 11.2 Lemma]): in other words V has alternating (resp. symmetric) rank $g = r$ (resp. $g = -r$). Suppose in particular that V is Kimura positive (resp. negative); then $r_{\wedge^i V} > 0$ (resp. $r_{\vee^i V} > 0$) for i even and Theorem 4.4 (resp. Theorem 5.3) applies. On the other hand, when i is odd, the condition $r_{\wedge^i V} < 0$ (resp. $r_{\vee^i V} > 0$) required by Theorem 4.4 (resp. Theorem 5.3) is not satisfied and we cannot apply our results.

It is known that the motive $h^1(A)$ of an abelian scheme of dimension d is Kimura negative of Kimura rank $2d$ (see [Ki1, Definitions 3.8 and 6.4] for the precise definitions). Suppose that $d = 2i \equiv 0 \bmod 4$, so that i is even and $r_{\vee^i V} > 0$. Since d is even, $\vee^{2d} h^1(A) \simeq h^{2d}(A) \simeq \mathbb{I}(-d)$ is the square of an invertible object. Theorem 5.3 applied to $V = h^1(A)$ implies the existence of canonical pieces

$$\ker\left(\partial^n_{d/2-1}\right) \subset \mathrm{Sym}^n \left(\vee^{d/2} h^1(A)\right) \otimes \vee^{d/2-1} h^1(A) \simeq \mathrm{Sym}^n \left(h^{d/2}(A)\right) \otimes h^{d/2-1}(A)$$

for every $n \geq 1$. Note that in [Ku] there is a different notation that is being used for the symmetric powers of a motive, namely $\Lambda^{\cdot}$, which is the authors' opinion can be slightly misleading.

The paper is organized as follows. In §2 we recall the needed results from [MS]. In §3 we discuss generalities on Laplace and Dirac operators in rigid and pseudo-abelian tensor categories, giving condition for the existence of kernel of Laplace operators and for the Dirac operators to be square roots of Laplace operators. We remark that the existence of kernels of Laplace operators is stated in [IS, §10] for the category of Chow motives; the authors are indebted with M. Spiess for providing them some notes on the topics. In §4 and §5 we use the Poincaré morphisms from §2 to define our Dirac operators on the alternating and symmetric powers and prove that they are indeed square roots of the Laplace operators; together with the result from §3 we deduce Theorems 4.4, 4.3 and 5.3. In §6 we discuss how the constructions behaves with respect to additive AU tensor functors which may not respect the associativity constraint, as needed for the realization functor R (see [Ku]); we also apply the results to the specific case of a quaternionic object, as needed for the construction of the motives of modular forms. The subsequent section is devoted to the computation of the realization of the motives of modular forms: the reader is strongly suggested to first give a look to this section as a motivation for the abstract constructions. We work with variations of Hodge structures as a target category, following ideas of [IS], but the same computations could be worked out for other realizations following the same pattern.

Throughout this paper we will always work in a $\mathbb{Q}$-linear rigid and pseudo-abelian ACU category $\mathcal{C}$ with unit object $\mathbb{I}$ and internal homs. We let $ev_X : X^\vee \otimes X \to \mathbb{I}$ be the evaluation and $ev_X^\tau := ev_X \circ \tau_{X,X} : X \otimes X^\vee \to \mathbb{I}$ be the opposite evaluation.

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2. Poincaré duality isomorphism

Given an object $V \in \mathcal{C}$, we may consider the associated alternating and symmetric algebras, denoted by $\wedge^{\cdot} V$ and, respectively, $\vee^{\cdot} V$. If $A.$ denotes one of these algebras, we have multiplication morphisms

$$\varphi_{i,j} : A_i \otimes A_j \to A_{i+j},$$

a data which is equivalent to

$$f_{i,j} : A_i \to \hom\left(A_j, A_{i+j}\right).$$

When $g \geq i$, we may consider the composite

$$D^{i,g} : A_i \stackrel{f_{i,g-i}}{\to} \hom\left(A_{g-i}, A_g\right) \stackrel{d}{\to} \hom\left(A_g^\vee, A_{g-i}^\vee\right) \stackrel{\alpha^{-1}}{\to} A_{g-i}^\vee \otimes A_g^{\vee\vee},$$

where $d : \hom(X, Y) \to \hom(Y^\vee, X^\vee)$ is the internal duality morphism and $\alpha : \hom(X, Y) \to Y \otimes X^\vee$ is the canonical morphism (see [MS, §2]). Working with the alternating or symmetric algebra of the dual $V^\vee$ yields a morphism

$$D^{i,g} : A_i^\vee \stackrel{f_{i,g-i}}{\to} \hom\left(A_{g-i}^\vee, A_g^\vee\right) \stackrel{d}{\to} \hom\left(A_g^{\vee\vee}, A_{g-i}^{\vee\vee}\right) \stackrel{\alpha^{-1}}{\to} A_{g-i}^{\vee\vee} \otimes A_g^{\vee\vee\vee}.$$

Employing the reflexivity morphism $i : X \to X^{\vee\vee}$ we can define (see [MS, (20)]):

$$D_{i,g} : A_i^\vee \stackrel{D^{i,g}}{\to} A_{g-i}^{\vee\vee} \otimes A_g^{\vee\vee\vee} \stackrel{i^{-1} \otimes i^{-1}}{\to} A_{g-i} \otimes A_g^\vee.$$

The following results have been proved in [MS, §5 and §6]. In order to state them, we first need to define the following morphisms:

$$\begin{aligned}
\varphi_{i,j}^{13} \quad &: \quad A_i \otimes B \otimes A_j \otimes C \stackrel{1 \otimes \tau_{B,A_j} \otimes 1}{\to} A_i \otimes A_j \otimes B \otimes C \stackrel{\varphi_{i,j} \otimes 1}{\to} A_{i+j} \otimes B \otimes C, \\
\varphi_{i,j}^{13} \quad &: \quad A_i^\vee \otimes B \otimes A_j^\vee \otimes C \stackrel{1 \otimes \tau_{B,A_j} \otimes 1}{\to} A_i^\vee \otimes A_j^\vee \otimes B \otimes C \stackrel{\varphi_{i,j} \otimes 1}{\to} A_{i+j}^\vee \otimes B^\vee \otimes C^\vee
\end{aligned}$$

and then

$$\begin{aligned}
\varphi_{g-i,i}^{13 \to A_g^\vee} \quad &: \quad A_{g-i} \otimes A_g^\vee \otimes A_i \otimes A_g^\vee \stackrel{\varphi_{g-i,i}^{13}}{\to} A_g \otimes A_g^\vee \otimes A_g^\vee \stackrel{ev_{A_g}^\tau \otimes 1}{\to} A_g^\vee, \\
\varphi_{g-i,i}^{13 \to A_g^{\vee\vee}} \quad &: \quad A_{g-i}^\vee \otimes A_g^{\vee\vee} \otimes A_i^\vee \otimes A_g^{\vee\vee} \stackrel{\varphi_{g-i,i}^{13}}{\to} A_g^\vee \otimes A_g^{\vee\vee} \otimes A_g^{\vee\vee} \stackrel{ev_{A_g^\vee}^\tau \otimes 1}{\to} A_g^{\vee\vee}.
\end{aligned}$$

In the following discussion we let $r := \operatorname{rank}(V) \in End(\mathbb{I})$.

Theorem 2.1. *The following diagrams are commutative, for every $g \geq i \geq 0$.*

(1)

$$\begin{tikzcd}
\wedge^i V \arrow[r, "D^{i,g}"'] \arrow[rrr, bend left, "(-1)^{i(g-i)}\binom{g}{g-i}^{-1}\binom{r-i}{g-i}"] & \wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee} \arrow[r, "D_{g-i,g} \otimes 1_{\wedge^g V^{\vee\vee}}"'] & \wedge^i V \otimes \wedge^g V^\vee \otimes \wedge^g V^{\vee\vee} \arrow[r, "1_{\wedge^i V} \otimes ev^{g,\tau}_{V^\vee,a}"'] & \wedge^i V
\end{tikzcd}$$

and

$$\begin{tikzcd}
\wedge^{g-i} V^\vee \arrow[r, "D_{g-i,g}"'] \arrow[rrr, bend left, "(-1)^{i(g-i)}\binom{g}{i}^{-1}\binom{r+i-g}{i}"] & \wedge^i V \otimes \wedge^g V^\vee \arrow[r, "D^{i,g} \otimes 1_{\wedge^g V^\vee}"'] & \wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^g V^\vee \arrow[r, "1_{\wedge^{g-i} V^\vee} \otimes ev^g_{V^\vee,a}"'] & \wedge^{g-i} V^\vee .
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
\wedge^i V \otimes \wedge^{g-i} V \arrow[r, "\varphi_{i,g-i}"] \arrow[d, "D^{i,g} \otimes D^{g-i,g}"] & \wedge^g V \arrow[d, "\binom{g}{g-i}^{-1}\binom{r-i}{g-i} \cdot i_{\wedge^g V}"] \\
\wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^i V^\vee \otimes \wedge^g V^{\vee\vee} \arrow[r, "\varphi^{13\to\wedge^g V^{\vee\vee}}_{g-i,i}"] & \wedge^g V^{\vee\vee},
\end{tikzcd}
\qquad
\begin{tikzcd}
\wedge^i V^\vee \otimes \wedge^{g-i} V^\vee \arrow[r, "\varphi_{i,g-i}"] \arrow[d, "D_{i,g} \otimes D_{g-i,g}"] & \wedge^g V^\vee \arrow[d, "\binom{g}{g-i}^{-1}\binom{r-i}{g-i}"] \\
\wedge^{g-i} V \otimes \wedge^g V^\vee \otimes \wedge^i V \otimes \wedge^g V^\vee \arrow[r, "\varphi^{13\to\wedge^g V^\vee}_{g-i,i}"] & \wedge^g V^\vee .
\end{tikzcd}$$

We say that V has *alternating rank* $g \in \mathbb{N}_{\geq 1}$, if $\wedge^g V$ is an invertible object and $\binom{r-i}{g-i}$ and $\binom{r+i-g}{i}$ are invertible for every $0 \leq i \leq g$. For example, when $End(\mathbb{I})$ is a field or $r \in \mathbb{Q}$, the second condition means that r is not a root of the polynomials $\binom{T-i}{g-i} \in \mathbb{Q}[T]$ and $\binom{T+i-g}{i} \in \mathbb{Q}[T]$ for every $0 \leq i \leq g$, i.e. that $r \neq i, i+1, ..., g-i-1$ and $r \neq g-i, g-i+1, ..., g-1$ for every $1 \leq i \leq g$.

We say that V has *strong alternating rank* $g \in \mathbb{N}_{\geq 1}$, if $\wedge^g V$ is an invertible object and $r = g$ (hence V has alternating rank g).

Corollary 2.2. *If V has alternating rank $g \in \mathbb{N}$ then, for every $0 \leq i \leq g$, the morphisms $D^{i,g}$, $D_{g-i,g}$, $D^{g-i,g}$ and $D_{i,g}$ are isomorphisms and the multiplication maps $\varphi^V_{i,g-i}$, $\varphi^V_{g-i,i}$, $\varphi^{V^\vee}_{i,g-i}$ and $\varphi^{V^\vee}_{g-i,i}$ are perfect pairings (meaning that the associate* hom *valued morphisms are isomorphisms). Furthermore, when V has strong alternating rank g, we have $\binom{r-i}{g-i} = \binom{r+i-g}{i} = 1$ in the commutative diagrams of Theorem 2.1.*

Proposition 2.3. *The following diagrams are commutative, when $\wedge^g V$ is invertible of rank $r_{\wedge^g V}$ (hence $r_{\wedge^g V} \in \{\pm 1\}$):*

$$\begin{tikzcd}
\wedge^i V \otimes \wedge^{g-i} V \otimes V \arrow[r, "\tau_{\wedge^i V \otimes \wedge^{g-i} V, V}"] \arrow[d, "\left(1_{\wedge^i V} \otimes \varphi_{g-i,1}, \left(1_{\wedge^{g-i} V} \otimes \varphi_{i,1}\right) \circ \left(\tau_{\wedge^i V, \wedge^{g-i} V} \otimes 1_V\right)\right)"] & V \otimes \wedge^i V \otimes \wedge^{g-i} V \arrow[d, "D^{1,g} \otimes \varphi_{i,g-i}"] \\
\wedge^i V \otimes \wedge^{g-i+1} V \oplus \wedge^{g-i} V \otimes \wedge^{i+1} V \arrow[d, "D^{i,g} \otimes D^{g-i+1,g} \oplus D^{g-i,g} \otimes D^{i+1,g}"] & \wedge^{g-1} V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^g V \arrow[d, "r_{\wedge^g V} g \binom{g}{g-i}^{-1}\binom{r-i}{g-i} \cdot 1_{\wedge^{g-1} V^\vee \otimes \wedge^g V^{\vee\vee}} \otimes i_{\wedge^g V}"] \\
\wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^{i-1} V^\vee \otimes \wedge^g V^{\vee\vee} \oplus \wedge^i V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^{g-i-1} V^\vee \otimes \wedge^g V^{\vee\vee} \arrow[r, "(-1)^{g-i} i \cdot \varphi^{13}_{g-i,i-1} \oplus (-1)^{i(g-i-1)} (g-i) \cdot \varphi^{13}_{i,g-i-1}"'] & \wedge^{g-1} V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\wedge^i V^\vee \otimes \wedge^{g-i} V^\vee \otimes V^\vee \arrow[r, "\tau_{\wedge^i V^\vee \otimes \wedge^{g-i} V^\vee, V^\vee}"] \arrow[d, "\left(1_{\wedge^i V^\vee} \otimes \varphi_{g-i,1}, \left(1_{\wedge^{g-i} V^\vee} \otimes \varphi_{i,1}\right) \circ \left(\tau_{\wedge^i V^\vee, \wedge^{g-i} V^\vee} \otimes 1_{V^\vee}\right)\right)"] & V^\vee \otimes \wedge^i V^\vee \otimes \wedge^{g-i} V^\vee \arrow[d, "D_{1,g} \otimes \varphi_{i,g-i}"] \\
\wedge^i V^\vee \otimes \wedge^{g-i+1} V^\vee \oplus \wedge^{g-i} V^\vee \otimes \wedge^{i+1} V^\vee \arrow[d, "D_{i,g} \otimes D_{g-i+1,g} \oplus D_{g-i,g} \otimes D_{i+1,g}"] & \wedge^{g-1} V \otimes \wedge^g V^\vee \otimes \wedge^g V^\vee \arrow[d, "r_{\wedge^g V} g \binom{g}{g-i}^{-1}\binom{r-i}{g-i} \cdot 1_{\wedge^{g-1} V^\vee \otimes \wedge^g V^\vee \otimes \wedge^g V^\vee}"] \\
\wedge^{g-i} V \otimes \wedge^g V^\vee \otimes \wedge^{i-1} V \otimes \wedge^g V^\vee \oplus \wedge^i V \otimes \wedge^g V^\vee \otimes \wedge^{g-i-1} V \otimes \wedge^g V^\vee \arrow[r, "(-1)^{g-i} i \cdot \varphi^{13}_{g-i,i-1} \oplus (-1)^{i(g-i-1)} (g-i) \cdot \varphi^{13}_{i,g-i-1}"'] & \wedge^{g-1} V \otimes \wedge^g V^\vee \otimes \wedge^g V^\vee .
\end{tikzcd}$$

Here are the analogue of the above results for the symmetric algebras.

Theorem 2.4. *The following diagrams are commutative, for every $g \geq i \geq 0$.*

(1)

$$\begin{tikzcd}
\vee^i V \arrow[r, "D^{i,g}"'] \arrow[rrr, bend left, "\binom{g}{g-i}^{-1}\binom{r+g-1}{g-i}"] & \vee^{g-i} V^\vee \otimes \vee^g V^{\vee\vee} \arrow[r, "D_{g-i,g} \otimes 1_{\vee^g V^{\vee\vee}}"'] & \vee^i V \otimes \vee^g V^\vee \otimes \vee^g V^{\vee\vee} \arrow[r, "1_{\vee^i V} \otimes ev^{g,\tau}_{V^\vee,s}"'] & \vee^i V
\end{tikzcd}$$

and

$$\begin{tikzcd}
\vee^{g-i} V^\vee \arrow[r, "D_{g-i,g}"'] \arrow[rrr, bend left, "\binom{g}{i}^{-1}\binom{r+g-1}{i}"] & \vee^i V \otimes \vee^g V^\vee \arrow[r, "D^{i,g} \otimes 1_{\vee^g V^\vee}"'] & \vee^{g-i} V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^g V^\vee \arrow[r, "1_{\vee^{g-i} V^\vee} \otimes ev^g_{V^\vee,s}"'] & \vee^{g-i} V^\vee .
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
\vee^i V \otimes \vee^{g-i} V \arrow[r, "\varphi_{i,g-i}"] \arrow[d, "D^{i,g} \otimes D^{g-i,g}"] & \vee^g V \arrow[d, "\binom{g}{g-i}^{-1}\binom{r+g-1}{g-i} \cdot i_{\vee^g V}"] \\
\vee^{g-i} V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^i V^\vee \otimes \vee^g V^{\vee\vee} \arrow[r, "\varphi^{13 \to \vee^g V^{\vee\vee}}_{g-i,i}"] & \vee^g V^{\vee\vee} ,
\end{tikzcd}
\qquad
\begin{tikzcd}
\vee^i V^\vee \otimes \vee^{g-i} V^\vee \arrow[r, "\varphi_{i,g-i}"] \arrow[d, "D_{i,g} \otimes D_{g-i,g}"] & \vee^g V^\vee \arrow[d, "\binom{g}{g-i}^{-1}\binom{r+g-1}{g-i}"] \\
\vee^{g-i} V \otimes \vee^g V^\vee \otimes \vee^i V \otimes \vee^g V^\vee \arrow[r, "\varphi^{13 \to \vee^g V^\vee}_{g-i,i}"] & \vee^g V^\vee .
\end{tikzcd}$$

We say that V has *symmetric rank* $g \in \mathbb{N}_{\geq 1}$, if $\vee^g V$ is an invertible object and $\binom{r+g-1}{g-i}$ and $\binom{r+g-1}{i}$ are invertible for every $0 \leq i \leq g$. For example, when $End(\mathbb{I})$ is a field or $r \in \mathbb{Q}$, the second condition means that r is not a root of the polynomials $\binom{T+g-1}{g-i} \in \mathbb{Q}[T]$ and $\binom{T+g-1}{i} \in \mathbb{Q}[T]$ for every $0 \leq i \leq g$, i.e. that $r \neq 1-g, 2-g, ..., -i$ and $r \neq 1-g, 2-g, ..., i-g$ for every $1 \leq i \leq g$.

We say that V has *strong symmetric rank* $g \in \mathbb{N}_{\geq 1}$, if $\vee^g V$ is an invertible object and $r = -g$ (hence V has symmetric rank g).

Corollary 2.5. *If V has symmetric rank $g \in \mathbb{N}$ then, for every $0 \leq i \leq g$, the morphisms $D^{i,g}$, $D_{g-i,g}$, $D^{g-i,g}$ and $D_{i,g}$ are isomorphisms and the multiplication maps $\varphi^V_{i,g-i}$, $\varphi^V_{g-i,i}$, $\varphi^{V^\vee}_{i,g-i}$ and $\varphi^{V^\vee}_{g-i,i}$ are perfect pairings (meaning that the associate* hom *valued morphisms are isomorphisms). Furthermore, when V has strong symmetric rank g, we have $\binom{r+g-1}{g-i} = (-1)^{g-i}$ and $\binom{r+g-1}{i} = (-1)^i$ in the commutative diagrams of Theorem 2.4.*

Proposition 2.6. *The following diagrams are commutative when $\vee^g V$ is invertible of rank $r_{\vee^g V}$ (hence $r_{\vee^g V} \in \{\pm 1\}$):*

$$\begin{tikzcd}
\vee^i V \otimes \vee^{g-i} V \otimes V \arrow[r, "\tau_{\vee^i V \otimes \vee^{g-i} V, V}"] \arrow[d, "\left(1_{\vee^i V} \otimes \varphi_{g-i,1}, \left(1_{\vee^{g-i} V} \otimes \varphi_{i,1}\right) \circ \left(\tau_{\vee^i V, \vee^{g-i} V} \otimes 1_V\right)\right)"] & V \otimes \vee^i V \otimes \vee^{g-i} V \arrow[d, "D^{1,g} \otimes \varphi_{i,g-i}"] \\
\vee^i V \otimes \vee^{g-i+1} V \oplus \vee^{g-i} V \otimes \vee^{i+1} V \arrow[d, "D^{i,g} \otimes D^{g-i+1,g} \oplus D^{g-i,g} \otimes D^{i+1,g}"] & \vee^{g-1} V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^g V \arrow[d, "r_{\vee^g V} g \binom{g}{g-i}^{-1}\binom{r+g-1}{g-i} \cdot 1_{\vee^{g-1} V^\vee \otimes \vee^g V^{\vee\vee}} \otimes i_{\vee^g V}"] \\
\vee^{g-i} V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^{i-1} V^\vee \otimes \vee^g V^{\vee\vee} \oplus \vee^i V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^{g-i-1} V^\vee \otimes \vee^g V^{\vee\vee} \arrow[r, "i \cdot \varphi^{13}_{g-i,i-1} \oplus (g-i) \cdot \varphi^{13}_{i,g-i-1}"'] & \vee^{g-1} V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^g V^{\vee\vee}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\vee^i V^\vee \otimes \vee^{g-i} V^\vee \otimes V^\vee \arrow[r, "\tau_{\vee^i V^\vee \otimes \vee^{g-i} V^\vee, V^\vee}"] \arrow[d, "\left(1_{\vee^i V^\vee} \otimes \varphi_{g-i,1}, \left(1_{\vee^{g-i} V^\vee} \otimes \varphi_{i,1}\right) \circ \left(\tau_{\vee^i V^\vee, \vee^{g-i} V^\vee} \otimes 1_{V^\vee}\right)\right)"] & V^\vee \otimes \vee^i V^\vee \otimes \vee^{g-i} V^\vee \arrow[d, "D_{1,g} \otimes \varphi_{i,g-i}"] \\
\vee^i V^\vee \otimes \vee^{g-i+1} V^\vee \oplus \vee^{g-i} V^\vee \otimes \vee^{i+1} V^\vee \arrow[d, "D_{i,g} \otimes D_{g-i+1,g} \oplus D_{g-i,g} \otimes D_{i+1,g}"] & \vee^{g-1} V \otimes \vee^g V^\vee \otimes \vee^g V^\vee \arrow[d, "r_{\vee^g V} g \binom{g}{g-i}^{-1}\binom{r+g-1}{g-i} \cdot 1_{\vee^{g-1} V^\vee \otimes \vee^g V^\vee \otimes \vee^g V^\vee}"] \\
\vee^{g-i} V \otimes \vee^g V^\vee \otimes \vee^{i-1} V \otimes \vee^g V^\vee \oplus \vee^i V \otimes \vee^g V^\vee \otimes \vee^{g-i-1} V \otimes \vee^g V^\vee \arrow[r, "i \cdot \varphi^{13}_{g-i,i-1} \oplus (g-i) \cdot \varphi^{13}_{i,g-i-1}"'] & \vee^{g-1} V \otimes \vee^g V^\vee \otimes \vee^g V^\vee .
\end{tikzcd}$$

3. Dirac and Laplace operators

If we are given $\psi : X \otimes Y \to Z$, we may consider

$$\partial_\psi^n := 1_{\otimes^{n-1}X} \otimes \psi : \otimes^n X \otimes Y \to \otimes^{n-1} X \otimes Z \text{ for } n \geq 1$$

and then we define

$$\begin{aligned} \partial_{\psi,a}^n &: \wedge^n X \otimes Y \overset{i_{X,a}^n \otimes 1_Y}{\to} \otimes^n X \otimes Y \overset{\partial_\psi^n}{\to} \otimes^{n-1} X \otimes Z \overset{p_{X,a}^{n-1} \otimes 1_Z}{\to} \wedge^{n-1} X \otimes Z, \\ \partial_{\psi,s}^n &: \vee^n X \otimes Y \overset{i_{X,s}^n \otimes 1_Y}{\to} \otimes^n X \otimes Y \overset{\partial_\psi^n}{\to} \otimes^{n-1} X \otimes Z \overset{p_{X,s}^{n-1} \otimes 1_Z}{\to} \vee^{n-1} X \otimes Z. \end{aligned}$$

Here we write $i_{X,*}^n$ and $p_{X,*}^n$ for the canonical injective and, respectively, surjective morphisms arising from the idempotent defining the alternating when $* = a$ and the symmetric when $* = s$ powers.

In particular, when $X = Y$, we have

$$\Delta_\psi^n = \partial_\psi^{n-1} = 1_{\otimes^{n-2}X} \otimes \psi : \otimes^n X \to \otimes^{n-2} X \otimes Z \text{ for } n \geq 2$$

inducing

$$\begin{aligned} \Delta_{\psi,a}^n &: \wedge^n X \overset{i_{X,a}^n}{\to} \otimes^n X \overset{\Delta_\psi^n}{\to} \otimes^{n-2} X \otimes Z \overset{p_{X,a}^{n-2} \otimes 1_Z}{\to} \wedge^{n-2} X \otimes Z, \\ \Delta_{\psi,s}^n &: \vee^n X \overset{i_{X,s}^n}{\to} \otimes^n X \overset{\Delta_\psi^n}{\to} \otimes^{n-2} X \otimes Z \overset{p_{X,s}^{n-2} \otimes 1_Z}{\to} \vee^{n-2} X \otimes Z. \end{aligned}$$

We may lift these morphisms to the tensor products as follows. Let ε (resp. 1) be the sign character (resp. trivial) character of the symmetric group and, if $\chi \in \{\varepsilon, 1\}$ and $R \subset S_k$ is any subset, define

$$e_R^\chi := \frac{1}{\# R} \sum\nolimits_{\delta \in R} \chi^{-1}(\delta)\, \delta \in \mathbb{Q}[S_k].$$

In particular, taking $R = S_k$ gives the idempotents $e_{X,a}^k := e_R^\varepsilon$ and $e_{X,s}^k := e_R^1$ defining the alternating and symmetric k-powers of any object X. We have that ∂_ψ^n (resp. Δ_ψ^n) is equivariant for the action of $S_{n-1} = S_{\{1,\ldots,n-1\}} \subset S_n$ (resp. $S_{n-2} = S_{\{1,\ldots,n-2\}} \subset S_n$). Furthermore, if we choose, for every $p \in \{1, \ldots, n\} =: I_n$ (resp. $(p,q) \in I_n \times I_n$ with $p \neq q$), elements $\delta_p^n \in S_n$ (resp. $\delta_{p,q}^n \in S_n$) such that $\delta_p^n(p) = n$ (resp. $\delta_{p,q}^{n-1,n}(p,q) = (n-1,n)$), then $R_{S_{n-1} \backslash S_n} := \left\{\delta_p^n : p \in I_n\right\}$ (resp. $R_{S_{n-2} \backslash S_n} := \left\{\delta_{p,q}^{n-1,n} : (p,q) \in I_n \times I_n, p \neq q\right\}$) is a set of coset representatives for $S_{n-1} \backslash S_n$ (resp. $S_{n-2} \backslash S_n$). Using these facts it is not difficult to check that, setting

$$\widetilde{\partial}_{\psi,*}^n := \partial_\psi^n \circ e_{R_{S_{n-1} \backslash S_n}}^\chi = \frac{1}{n} \sum\nolimits_{p=1}^n \chi^{-1}\left(\delta_p^n\right) \cdot \left(1_{\otimes^{n-1}X} \otimes \psi\right) \circ \left(\delta_p^n \otimes 1_Y\right), \tag{5}$$

$$\widetilde{\Delta}_{\psi,*}^n := \Delta_\psi^n \circ e_{R_{S_{n-2} \backslash S_n}}^\chi = \frac{1}{n(n-1)} \sum\nolimits_{p,q \in I_n : p \neq q} \chi^{-1}\left(\delta_{p,q}^{n-1,n}\right) \cdot \left(1_{\otimes^{n-2}X} \otimes \psi\right) \circ \delta_{p,q}^{n-1,n}, \tag{6}$$

where $* \in \{a, s\}$ depending, respectively, on whether χ is ε or 1. This gives morphisms making the following diagrams commutative:

$$\begin{tikzcd} \otimes^n X \otimes Y \arrow[r, "\widetilde{\partial}_{\psi,a}^n"] \arrow[d, "p_{X,a}^n \otimes 1_Y"'] & \otimes^{n-1} X \otimes Z \arrow[d, "p_{X,a}^{n-1} \otimes 1_Z"] \\ \wedge^n X \otimes Y \arrow[r, "\partial_{\psi,a}^n"] & \wedge^{n-1} X \otimes Z, \end{tikzcd} \qquad \begin{tikzcd} \otimes^n X \otimes Y \arrow[r, "\widetilde{\partial}_{\psi,a}^n"] \arrow[d, "p_{X,s}^n \otimes 1_Y"'] & \otimes^{n-1} X \otimes Z \arrow[d, "p_{X,s}^{n-1} \otimes 1_Z"] \\ \vee^n X \otimes Y \arrow[r, "\partial_{\psi,s}^n"] & \vee^{n-1} X \otimes Z, \end{tikzcd} \tag{7}$$

$$\begin{tikzcd} \otimes^n X \arrow[r, "\widetilde{\Delta}_{\psi,a}^n"] \arrow[d, "p_{X,a}^n"'] & \otimes^{n-2} X \otimes Z \arrow[d, "p_{X,a}^{n-2} \otimes 1_Z"] \\ \wedge^n X \arrow[r, "\Delta_{\psi,a}^n"] & \wedge^{n-2} X \otimes Z, \end{tikzcd} \qquad \begin{tikzcd} \otimes^n X \arrow[r, "\widetilde{\Delta}_{\psi,a}^n"] \arrow[d, "p_{X,s}^n"'] & \otimes^{n-2} X \otimes Z \arrow[d, "p_{X,s}^{n-2} \otimes 1_Z"] \\ \vee^n X \arrow[r, "\Delta_{\psi,s}^n"] & \vee^{n-2} X \otimes Z. \end{tikzcd}$$

When ψ is alternating or symmetric, we can refine (6) as follows.

Lemma 3.1. *Suppose that $\psi : X \otimes X \to Z$ is such that $\psi \circ \tau_{X,X} = -\psi$ (resp. $\psi \circ \tau_{X,X} = \psi$). Then $\Delta^n_{\psi,s} = 0$ (resp. $\Delta^n_{\psi,a} = 0$) and $\Delta^n_{\psi,a}$ (resp. $\Delta^n_{\psi,s}$) is induced by*

$$\widehat{\Delta}^n_{\psi,*} := \frac{2}{n\,(n-1)} \sum\nolimits_{p,q \in I_n : p<q} \chi^{-1}\left(\delta^{n-1,n}_{p,q}\right) \cdot \left(1_{\otimes^{n-2}X} \otimes \psi\right) \circ \delta^{n-1,n}_{p,q}$$

where $\chi = \varepsilon$ (resp. $\chi = 1$), $* = a$ (resp. $* = s$) and $\delta^{n-1,n}_{p,q}\,(p,q) = (n-1,n)$.

Proof. The proof, based on (6) and the subsequent Remark 3.2, is left to the reader. □

Remark 3.2. *Suppose that we are given actions of S_n on A and of S_{n-2} on B and that $f : A \to B$ is an S_{n-2}-equivariant map, for an integer $n \geq 2$. Then we have, setting $\tau_{n-1,n} := (n-1,n)$,*

$$\begin{aligned} e^{\chi}_{S_{n-2}} \circ f \circ e^{\chi}_{R_{S_{n-2}\setminus S_n}} &:= e^{\chi}_{S_{n-2}} \circ \frac{1}{n\,(n-1)} \sum\nolimits_{p,q \in I_n : p \neq q} \chi^{-1}\left(\delta^{n-1,n}_{p,q}\right) \cdot f \circ \delta^{n-1,n}_{p,q} \\ &= e^{\chi}_{S_{n-2}} \circ \frac{1}{n\,(n-1)} \sum\nolimits_{p,q \in I_n : p<q} \chi^{-1}\left(\delta^{n-1,n}_{p,q}\right) \cdot \left(f + \chi^{-1}\left(\tau_{n-1,n}\right) \cdot f \circ \tau_{n-1,n}\right) \circ \delta^{n-1,n}_{p,q}. \end{aligned}$$

Suppose now that we are given $\psi_1 : X \otimes Y \to Z$ and $\psi_2 : X \otimes Z \to Y \otimes W$ and $\psi : X \otimes X \to W$. They induce

$$\begin{aligned} &\wedge^n X \otimes Y \overset{\partial^n_{\psi_1,a}}{\to} \wedge^{n-1} X \otimes Z \overset{\partial^{n-1}_{\psi_2,a}}{\to} \wedge^{n-2} X \otimes Y \otimes W, \\ &\vee^n X \otimes Y \overset{\partial^n_{\psi_1,s}}{\to} \vee^{n-1} X \otimes Z \overset{\partial^{n-1}_{\psi_2,s}}{\to} \vee^{n-2} X \otimes Y \otimes W \end{aligned}$$

and

$$\begin{aligned} \Delta^n_{\psi,a} &: \wedge^n X \to \wedge^{n-2} X \otimes W, \\ \Delta^n_{\psi,s} &: \vee^n X \to \vee^{n-2} X \otimes W. \end{aligned}$$

Lemma 3.3. *Suppose that $\psi : X \otimes X \to W$ is such that $\psi \circ \tau_{X,X} = \nu_* \cdot \psi$, where $\nu_a := -1$, $\nu_s := 1$ and $* \in \{a, s\}$, and that, for some $\rho \in End\,(\mathbb{I})$, the following diagram is commutative:*

$$\begin{tikzcd} X \otimes X \otimes Y \arrow[r, "(1_X \otimes \psi_1, (1_X \otimes \psi_1) \circ (\tau_{X,X} \otimes 1_Y))"] \arrow[d, "\psi \otimes 1_Y"'] & X \otimes Z \oplus X \otimes Z \arrow[d, "\psi_2 \oplus \nu_* \cdot \psi_2"] \\ W \otimes Y \arrow[r, "\rho \cdot \tau_{W,Y}"] & Y \otimes W. \end{tikzcd}$$

Then, when $* = a$, the following diagram is commutative

$$\begin{tikzcd} \wedge^n X \otimes Y \arrow[r, "\partial^n_{\psi_1,a}"] \arrow[d, "\Delta^n_{\psi,a} \otimes 1_Y"'] & \wedge^{n-1} X \otimes Z \arrow[d, "\partial^{n-1}_{\psi_2,a}"] \\ \wedge^{n-2} X \otimes W \otimes Y \arrow[r, "\frac{\rho}{2} \cdot 1_{\wedge^{n-2}X} \otimes \tau_{W,Y}"] & \wedge^{n-2} X \otimes Y \otimes W \end{tikzcd}$$

and, when $* = s$, the following diagram is commutative:

$$\begin{tikzcd} \vee^n X \otimes Y \arrow[r, "\partial^n_{\psi_1,s}"] \arrow[d, "\Delta^n_{\psi,s} \otimes 1_Y"'] & \vee^{n-1} X \otimes Z \arrow[d, "\partial^{n-1}_{\psi_2,s}"] \\ \vee^{n-2} X \otimes W \otimes Y \arrow[r, "\frac{\rho}{2} \cdot 1_{\vee^{n-2}X} \otimes \tau_{W,Y}"] & \vee^{n-2} X \otimes Y \otimes W. \end{tikzcd}$$

Proof. We compute, using the notations in (5),

$$\begin{aligned}
\widetilde{\partial}^{n-1}_{\psi_2,*}\circ\widetilde{\partial}^{n}_{\psi_1,*} &= \frac{1}{n\left(n-1\right)}\sum\nolimits_{\substack{q=1,\dots,n\\ p=1,\dots,n-1}}\chi^{-1}\left(\delta^{n-1}_p\delta^n_q\right)\cdot\left(1_{\otimes^{n-2}X}\otimes\psi_2\right)\circ\left(\delta^{n-1}_p\otimes 1_Z\right)\\
&\quad\circ\left(1_{\otimes^{n-1}X}\otimes\psi_1\right)\circ\left(\delta^n_q\otimes 1_Y\right)\\
&= \frac{1}{n\left(n-1\right)}\sum\nolimits_{\substack{q=1,\dots,n\\ p=1,\dots,n-1}}\chi^{-1}\left(\delta^{n-1}_p\delta^n_q\right)\cdot\left(1_{\otimes^{n-2}X}\otimes\psi_2\right)\circ\left(1_{\otimes^{n-1}X}\otimes\psi_1\right)\\
&\quad\circ\left(\delta^{n-1}_p\otimes 1_{X\otimes Y}\right)\circ\left(\delta^n_q\otimes 1_Y\right).
\end{aligned}\tag{8}$$

Here $\delta^{n-1}_p\otimes 1_{X\otimes Y}$ acts on $\otimes^n X\otimes Y$ as $\delta^{n-1}_p\otimes 1_Y$, where now $\delta^{n-1}_p\in S_{n-1}=S_{\{1,\dots,n-1\}}\subset S_n$ is viewed in S_n, so that $\left(\delta^{n-1}_p\otimes 1_{X\otimes Y}\right)\circ\left(\delta^n_q\otimes 1_Y\right)=\delta^{n-1}_p\delta^n_q\otimes 1_Y$. We now remark that we may choose δ^n_q so that $\delta^n_q\left(p\right)=p$ if $p\in\{1,\dots,n-1\}-\{q\}$ and then we find

$$\delta^{n-1}_p\delta^n_q\left(p,q\right)=\delta^{n-1}_p\left(p,n\right)=\left(n-1,n\right),\text{ if }p\in\{1,\dots,n-1\}-\{q\}.$$

On the other hand, we may further assume that $\delta^n_q\left(n\right)=q$ (with $\delta^n_q=\left(q,n\right)$ both the imposed conditions are indeed satisfied). Then we find

$$\delta^{n-1}_q\delta^n_q\left(n,q\right)=\delta^{n-1}_q\left(q,n\right)=\left(n-1,n\right),\text{ if }q\in\{1,\dots,n-1\}.$$

Summarizing, setting $\delta^{n-1,n}_{p,q}:=\delta^{n-1}_p\delta^n_q$ if $p\in\{1,\dots,n-1\}-\{q\}$ and $\delta^{n-1,n}_{n,q}:=\delta^{n-1}_q\delta^n_q$ if $q\in\{1,\dots,n-1\}$, we see that

$$\{\left(p,q\right)\in I_n\times I_n,p\neq q\}=\{\left(p,q\right):p\in I_{n-1}-\{q\}\}\sqcup\{\left(n,q\right):q\in I_{n-1}\}$$

and then, since $\delta^{n-1,n}_{p,q}\left(p,q\right)=\left(n-1,n\right)$, we have

$$R_{S_{n-2}\setminus S_n}=\left\{\delta^{n-1,n}_{p,q}:\left(p,q\right)\in I_n\times I_n,p\neq q\right\}.$$

Setting $f:=\left(1_{\otimes^{n-2}X}\otimes\psi_2\right)\circ\left(1_{\otimes^{n-1}X}\otimes\psi_1\right)$ it follows from (8) and the above discussion that we have

$$\widetilde{\partial}^{n-1}_{\psi_2,*}\circ\widetilde{\partial}^{n}_{\psi_1,*}=\frac{1}{n\left(n-1\right)}\sum\nolimits_{p,q\in I_n:p\neq q}\chi^{-1}\left(\delta^{n-1,n}_{p,q}\right)\cdot f\circ\left(\delta^{n-1,n}_{p,q}\otimes 1_Y\right).\tag{9}$$

Noticing that f is S_{n-2}-equivariant we may apply Remark 3.2 to get

$$\begin{aligned}
&e^{\chi}_{S_{n-2}}\circ\widetilde{\partial}^{n-1}_{\psi_2,*}\circ\widetilde{\partial}^{n}_{\psi_1,*}\\
&=e^{\chi}_{S_{n-2}}\circ\frac{1}{n\left(n-1\right)}\sum\nolimits_{p,q\in I_n:p<q}\chi^{-1}\left(\delta^{n-1,n}_{p,q}\right)\cdot\left(f+\chi^{-1}\left(\tau_{n-1,n}\right)\cdot f\circ\tau_{n-1,n}\right)\circ\left(\delta^{n-1,n}_{p,q}\otimes 1_Y\right).
\end{aligned}$$

We now remark that the relation

$$\psi_2\circ\left(1_X\otimes\psi_1\right)+\nu_*\cdot\psi_2\circ\left(1_X\otimes\psi_1\right)\circ\left(\tau_{X,X}\otimes 1_Y\right)=\rho\cdot\tau_{W,Y}\circ\left(\psi\otimes 1_Y\right)$$

gives, thanks to $\nu_*=\chi^{-1}\left(\tau_{n-1,n}\right)$,

$$\begin{aligned}
f+\chi^{-1}\left(\tau_{n-1,n}\right)\cdot f\circ\tau_{n-1,n} &= \left(1_{\otimes^{n-2}X}\otimes\psi_2\right)\circ\left(1_{\otimes^{n-1}X}\otimes\psi_1\right)\\
&\quad+\nu_*\cdot\left(1_{\otimes^{n-2}X}\otimes\psi_2\right)\circ\left(1_{\otimes^{n-1}X}\otimes\psi_1\right)\circ\left(\tau_{n-1,n}\otimes 1_Y\right)\\
&= \rho\cdot\left(1_{\otimes^{n-2}X}\otimes\tau_{W,Y}\right)\circ\left(1_{\otimes^{n-2}X}\otimes\psi\otimes 1_Y\right).
\end{aligned}$$

Hence (9) gives

$$\begin{aligned}
e^{\chi}_{S_{n-2}}\circ\widetilde{\partial}^{n-1}_{\psi_2,*}\circ\widetilde{\partial}^{n}_{\psi_1,*} &= e^{\chi}_{S_{n-2}}\circ\left(1_{\otimes^{n-2}X}\otimes\tau_{W,Y}\right)\\
&\quad\circ\frac{\rho}{n\left(n-1\right)}\sum\nolimits_{p,q\in I_n:p<q}\chi^{-1}\left(\delta^{n-1,n}_{p,q}\right)\cdot\left(1_{\otimes^{n-2}X}\otimes\psi\otimes 1_Y\right)\circ\left(\delta^{n-1,n}_{p,q}\otimes 1_Y\right).
\end{aligned}\tag{10}$$

We have $e^{\chi}_{S_{n-2}}=e^{n-2}_{X,*}\otimes 1_T=\left(p^{n-2}_{X,*}\otimes 1_T\right)\circ\left(i^{n-2}_{X,*}\otimes 1_T\right)$ and $i^{n-2}_{X,*}\otimes 1_T$ is a monomorphism (10), where $T=Y\otimes W$ on the left hand side while $T=W\otimes Y$ on the right hand side of (10). Hence (10) gives, with the notations of Lemma 3.1,

$$\begin{aligned}
&2\cdot\left(p^{n-2}_{X,*}\otimes 1_{Y\otimes W}\right)\circ\widetilde{\partial}^{n-1}_{\psi_2,*}\circ\widetilde{\partial}^{n}_{\psi_1,*}=\rho\cdot\left(p^{n-2}_{X,*}\otimes 1_{W\otimes Y}\right)\circ\left(1_{\otimes^{n-2}X}\otimes\tau_{W,Y}\right)\circ\left(\widehat{\Delta}^{n}_{\psi,*}\otimes 1_Y\right)\\
&=\rho\cdot\left(1_{n-2}\otimes\tau_{W,Y}\right)\circ\left(p^{n-2}_{X,*}\otimes 1_{W\otimes Y}\right)\circ\left(\widehat{\Delta}^{n}_{\psi,*}\otimes 1_Y\right),
\end{aligned}$$

where $1_{n-2} = 1_{\wedge^{n-2}X}$ when $* = a$ or, respectively, $1_{n-2} = 1_{\vee^{n-2}X}$ when $* = s$. Now the claim follows from (7), which gives

$$\left(p^{n-2}_{X,*} \otimes 1_{Y\otimes W}\right) \circ \widetilde{\partial}^{n-1}_{\psi_2,*} \circ \widetilde{\partial}^{n}_{\psi_1,*} = \partial^{n-1}_{\psi_2,*} \circ \left(p^{n-1}_{X,*} \otimes 1_Z\right) \circ \widetilde{\partial}^{n}_{\psi_1,*} = \partial^{n-1}_{\psi_2,*} \circ \partial^{n}_{\psi_1,*} \circ \left(p^{n}_{X,*} \otimes 1_Y\right),$$

and Lemma 3.1, which gives

$$\left(p^{n-2}_{X,*} \otimes 1_{W\otimes Y}\right) \circ \left(\widehat{\Delta}^{n}_{\psi,*} \otimes 1_Y\right) = \left(\Delta^{n}_{\psi,*} \otimes 1_Y\right) \circ \left(p^{n}_{X,*} \otimes 1_Y\right),$$

because $p^{n}_{X,*} \otimes 1_Y$ is an epimorphism. □

Suppose now that we are given a perfect pairing $\psi : X \otimes X \to \mathbb{I}$, meaning that the associated hom valued morphism $f_\psi : X \to X^\vee$ is an isomorphism. Then (X, ψ) is a dual pair for X and we have the Casimir element $C_\psi : \mathbb{I} \to X \otimes X$. It follows from well known properties of the Casimir element that we have the following commutative diagrams:

$$1_X : X \overset{C_\psi \otimes 1_X}{\to} X \otimes X \otimes X \overset{1_X \otimes \psi}{\to} X, \tag{11}$$

$$1_X : X \overset{1_X \otimes C_\psi}{\to} X \otimes X \otimes X \overset{\psi \otimes 1_X}{\to} X. \tag{12}$$

Suppose that we have $\psi \circ \tau_{X,X} = \chi(\tau_{X,X}) \cdot \psi$, where $\chi \in \{1, \varepsilon\}^2$. Recall that we write $r_X = \operatorname{rank}(X) := \psi \circ \tau_{X,X} \circ C_\psi$. Then we have $r_X = \chi(\tau_{X,X}) \cdot \psi \circ C_\psi$, implying that the following diagram is commutative:

$$\chi(\tau_{X,X})\, r_X : \mathbb{I} \overset{C_\psi}{\to} X \otimes X \overset{\psi}{\to} \mathbb{I}. \tag{13}$$

We may consider

$$C^n_\psi := 1_{\otimes^n X} \otimes C_\psi : \otimes^n X \to \otimes^{n+2} X \text{ for } n \geq 0$$

and then we define

$$\begin{aligned} C^n_{\psi,a} &: \wedge^n X \overset{i^n_{X,a}}{\to} \otimes^n X \overset{C^n_\psi}{\to} \otimes^{n+2} X \overset{p^{n+2}_{X,a}}{\to} \wedge^{n+2} X, \\ C^n_{\psi,s} &: \vee^n X \overset{i^n_{X,s}}{\to} \otimes^n X \overset{C^n_\psi}{\to} \otimes^{n+2} X \overset{p^{n+2}_{X,s}}{\to} \vee^{n+2} X. \end{aligned}$$

Since C^n_ψ is S_n-equivariant, the following diagrams are commutative:

$$\begin{array}{ccccccc} \otimes^n X & \xrightarrow{C^n_\psi} & \otimes^{n+2} X & \qquad & \otimes^n X & \xrightarrow{C^n_\psi} & \otimes^{n+2} X \\ \downarrow p^n_{X,a} & & \downarrow p^{n+2}_{X,a} & & \downarrow p^n_{X,s} & & \downarrow p^{n+2}_{X,s} \\ \wedge^n X & \xrightarrow{C^n_{\psi,a}} & \wedge^{n+2} X, & & \vee^n X & \xrightarrow{C^n_{\psi,s}} & \vee^{n+2} X. \end{array} \tag{14}$$

Lemma 3.4. *Suppose that $\psi : X \otimes X \to \mathbb{I}$ is a perfect pairing such that $\psi \circ \tau_{X,X} = \nu_* \cdot \psi$, where $\nu_a := -1$, $\nu_s := 1$ and $* \in \{a, s\}$. Then we have the formulas $\Delta^2_{\psi,*} \circ C^0_{\psi,*} = \nu_* r_X$, $3\Delta^3_{\psi,*} \circ C^1_{\psi,} = (2 + \nu_* r_X) \cdot 1_X$ and, for every $n \geq 2$,*

$$\frac{(n+2)(n+1)}{2} \cdot \Delta^{n+2}_{\psi,*} \circ C^n_{\psi,*} - \frac{n(n-1)}{2} \cdot C^{n-2}_{\psi,*} \circ \Delta^n_{\psi,*} = (2n + \nu_* r_X) \cdot 1_{(*)^n X},$$

where $(*)^n X := \wedge^n X$ for $* = a$, $(*)^n X := \vee^n X$ for $* = s$ and $r_X := \operatorname{rank}(X)$.

[2] We remark that, assuming 2 is invertible in $Hom(X \otimes X, \mathbb{I})$, we may always write ψ as the direct sum of its alternating and symmetric part, defined respectively by the formulas $\psi_a := \frac{\psi - \psi \circ \tau_{X,X}}{2}$ and $\psi_s := \frac{\psi + \psi \circ \tau_{X,X}}{2}$. This means that $\psi = \psi_a \oplus \psi_s$, up to the identification $Hom(X \otimes X, \mathbb{I}) = Hom(\wedge^2 X, \mathbb{I}) \oplus Hom(\vee^2 X, \mathbb{I})$ and the above assumption is always achieved by ψ_a and ψ_s.

Proof. We have, employing the notations in Lemma 3.1,

$$\begin{aligned}
&\frac{(n+2)(n+1)}{2}\cdot\widehat{\Delta}_{\psi,*}^{n+2}\circ C_{\psi}^{n}\\
&\quad=\textstyle\sum_{p,q\in I_{n+2}:p<q}\chi^{-1}\left(\delta_{p,q}^{n+1,n+2}\right)\cdot\left(1_{\otimes^{n}X}\otimes\psi\right)\circ\delta_{p,q}^{n+1,n+2}\circ\left(1_{\otimes^{n}X}\otimes C_{\psi}\right), \qquad (15)\\
&\frac{n(n-1)}{2}\cdot C_{\psi}^{n-2}\circ\widehat{\Delta}_{\psi,*}^{n}\\
&\quad=\textstyle\sum_{p,q\in I_{n}:p<q}\chi^{-1}\left(\delta_{p,q}^{n-1,n}\right)\cdot\left(1_{\otimes^{n-2}X}\otimes C_{\psi}\right)\circ\left(1_{\otimes^{n-2}X}\otimes\psi\right)\circ\delta_{p,q}^{n-1,n}. \qquad (16)
\end{aligned}$$

We claim that we have $\widehat{\Delta}_{\psi,*}^{2}\circ C_{\psi}^{0}=\nu_{*}r_{X}$, $3\widehat{\Delta}_{\psi,*}^{3}\circ C_{\psi}^{1}=(2+\nu_{*}r_{X})\cdot 1_{X}$ and, for every $n\geq 2$,

$$\frac{(n+2)(n+1)}{2}\cdot\widehat{\Delta}_{\psi,*}^{n+2}\circ C_{\psi}^{n}-\frac{n(n-1)}{2}\cdot C_{\psi}^{n-2}\circ\widehat{\Delta}_{\psi,*}^{n}=2n\cdot e_{R_{S_{n-1}\backslash S_{n}}}^{\chi}+\nu_{*}r_{X}\cdot 1_{\otimes^{n}X}.$$

It will follow from Lemma 3.1 and (14) that we have

$$\Delta_{\psi,*}^{2}\circ C_{\psi,*}^{0}=\nu_{*}r_{X},\, 3\Delta_{\psi,*}^{3}\circ C_{\psi,*}^{1}=(2+\nu_{*}r_{X})\cdot 1_{X}$$

and, for every $n\geq 2$,

$$\begin{aligned}
&\left(\frac{(n+2)(n+1)}{2}\cdot\Delta_{\psi,*}^{n+2}\circ C_{\psi,*}^{n}-\frac{n(n-1)}{2}\cdot C_{\psi,*}^{n-2}\circ\Delta_{\psi,*}^{n}\right)\circ p_{X,*}^{n}\\
&=p_{X,*}^{n}\circ\left(\frac{(n+2)(n+1)}{2}\cdot\widehat{\Delta}_{\psi,*}^{n+2}\circ C_{\psi}^{n}-\frac{n(n-1)}{2}\cdot C_{\psi}^{n-2}\circ\widehat{\Delta}_{\psi,*}^{n}\right)\\
&=2n\cdot p_{X,*}^{n}\circ e_{R_{S_{n-1}\backslash S_{n}}}^{\chi}+\nu_{*}r_{X}\cdot p_{X,*}^{n}=(2n+\nu_{*}r_{X})\cdot p_{X,*_{\chi}}^{n}.
\end{aligned}$$

Here in the last equality we have employed the relation $p_{X,*_{\chi}}^{n}\circ e_{R_{S_{n-1}\backslash S_{n}}}^{\chi}=p_{X,*}^{n}$, which is proved noticing that, since $i_{X,*}^{n}$ is a monomorphism, it is equivalent to $e_{X,*}^{n}\circ e_{R_{S_{n-1}\backslash S_{n}}}^{\chi}=e_{X,*}^{n}$; this last relation follows from the relations $e_{G}^{\chi}=e_{G}^{\chi}e_{H}^{\chi}$ and $e_{H}^{\chi}e_{R_{H\backslash G}}^{\chi}=e_{G}^{\chi}$, implying $e_{G}^{\chi}e_{R_{H\backslash G}}^{\chi}=e_{G}^{\chi}e_{H}^{\chi}e_{R_{H\backslash G}}^{\chi}=e_{G}^{\chi}e_{G}^{\chi}=e_{G}^{\chi}$, which can be easily checked in the group algebras. Then the claim will follow from the fact that $p_{X,*}^{n}$ is an epimorphism.

When $n=0$ in (15), we have $C_{\psi}^{0}=C_{\psi}$, $\widehat{\Delta}_{\psi,*}^{2}=\psi$ and the equality $\widehat{\Delta}_{\psi,*}^{2}\circ C_{\psi}^{0}=\nu_{*}r_{X}$ follows from (13). When $n=1$ in (15), we have

$$\begin{aligned}
3\widehat{\Delta}_{\psi,*}^{3}\circ C_{\psi}^{1} \quad = \quad & \chi^{-1}\left(\tau_{(123)}\right)\cdot(1_{X}\otimes\psi)\circ\tau_{(123)}\circ(1_{X}\otimes C_{\psi})\\
&+\chi^{-1}\left(\tau_{(12)}\right)\cdot(1_{X}\otimes\psi)\circ\tau_{(12)}\circ(1_{X}\otimes C_{\psi})\\
&+(1_{X}\otimes\psi)\circ(1_{X}\otimes C_{\psi}), \qquad (17)
\end{aligned}$$

because we may take $\delta_{1,2}^{2,3}=\tau_{(123)}$, $\delta_{1,3}^{2,3}=\tau_{(12)}$ and $\delta_{2,3}^{2,3}=1$, where τ_{σ} denotes the morphism attached to the permutation σ. We have $\tau_{(123)}=\tau_{X\otimes X,X}$ and $(\psi\otimes 1_{X})\circ(1_{X}\otimes C_{\psi})=1_{X}$ by (12). Hence we deduce the equality

$$\begin{aligned}
&\chi^{-1}\left(\tau_{(123)}\right)\cdot(1_{X}\otimes\psi)\circ\tau_{(123)}\circ(1_{X}\otimes C_{\psi})=(1_{X}\otimes\psi)\circ\tau_{X\otimes X,X}\circ(1_{X}\otimes C_{\psi})\\
=\;&(\psi\otimes 1_{X})\circ(1_{X}\otimes C_{\psi})=1_{X}. \qquad (18)
\end{aligned}$$

Consider the following diagram:

$$\begin{tikzcd}
 & X\otimes X\otimes X \arrow[d, "\tau_{X,X\otimes X}"] & \\
X \arrow[ur, bend left, "1_{X}\otimes C_{\psi}"] \arrow[r, "C_{\psi}\otimes 1_{X}"] \arrow[r, phantom, shift left=6ex, "(\tau)"] & X\otimes X\otimes X \arrow[r, "\nu_{*}\cdot 1_{X}\otimes\psi"] \arrow[d, "1_{X}\otimes\tau_{X,X}"'] \arrow[r, phantom, shift right=6ex, "(\tau)"] & X \\
 & X\otimes X\otimes X \arrow[ur, bend right, "1_{X}\otimes\psi"'] &
\end{tikzcd}$$

The region (A) is commutative thanks to our assumption $\psi\circ\tau_{X,X}=\nu_{*}\cdot\psi$. Noticing that $\tau_{(12)}=(1_{X}\otimes\tau_{X,X})\circ\tau_{X,X\otimes X}$ and that $(1_{X}\otimes\psi)\circ(C_{\psi}\otimes 1_{X})=1_{X}$ by (11), we deduce the equality:

$$\begin{aligned}
&\chi^{-1}\left(\tau_{(12)}\right)\cdot(1_{X}\otimes\psi)\circ\tau_{(12)}\circ(1_{X}\otimes C_{\psi})=\chi^{-1}\left(\tau_{(12)}\right)\cdot(1_{X}\otimes\psi)\circ(1\otimes\tau_{X,X})\circ\tau_{X,X\otimes X}\circ(1_{X}\otimes C_{\psi})\\
&\quad=\chi^{-1}\left(\tau_{(12)}\right)\nu_{*}\cdot(1_{X}\otimes\psi)\circ(C_{\psi}\otimes 1_{X})=1_{X}. \qquad (19)
\end{aligned}$$

Inserting (18), (19) and (13) in (17) gives $3\widehat{\Delta}^3_{\psi,*}\circ C^1_\psi=(2+\nu_* r_X)\cdot 1_X$.

Suppose now that $n\geq 2$. We remark that we have

$$\begin{aligned}\{(p,q)\in I_{n+2}\times I_{n+2}:p<q\} \quad &= \quad \{(p,q)\in I_n\times I_n : p<q\}\\ &\qquad \sqcup (I_n\times\{n+1\})\sqcup(I_n\times\{n+2\})\sqcup\{(n+1,n+2)\}\,. \end{aligned}\tag{20}$$

We may assume that we are given our choice of $\delta^{n-1,n}_{p,q}\in S_n$ and we choose the elements $\delta^{n+1,n+2}_{p,q}\in S_{n+2}$ as follows. First of all we view $S_n\subset S_{n+2}$ in the natural way, via $I_n\subset I_{n+2}$, and we choose elements $\delta^n_p\in S_n$ such that $\delta^n_p(p)=n$. If $(p,q)\in I_n\times I_n$ and $p<q$, we set $\delta^{n+1,n+2}_{p,q}:=\tau_{(n-1,n+1)(n,n+2)}\circ\delta^{n-1,n}_{p,q}$ and then we define $\delta^{n+1,n+2}_{p,n+1}:=\tau_{(n,n+1,n+2)}\circ\delta^n_p$, $\delta^{n+1,n+2}_{p,n+2}:=\tau_{(n,n+1)}\circ\delta^n_p$ and $\delta^{n+1,n+2}_{n+1,n+2}=1$, noticing that, in every case, we have the required relation $\delta^{n+1,n+2}_{p,q}(p,q)=(n+1,n+2)$ satisfied. Thanks to (20), we may rewrite (15) as follows:

$$\begin{aligned}\frac{(n+2)\,(n+1)}{2}\cdot\widehat{\Delta}^{n+2}_{\psi,*}\circ C^n_\psi &= \textstyle\sum_{p,q\in I_n:p<q}\chi^{-1}\left(\delta^{n+1,n+2}_{p,q}\right)\cdot(1_{\otimes^n X}\otimes\psi)\circ\delta^{n+1,n+2}_{p,q}\circ(1_{\otimes^n X}\otimes C_\psi)\\ &\quad+\textstyle\sum_{p\in I_n}\chi^{-1}\left(\delta^{n+1,n+2}_{p,n+1}\right)\cdot(1_{\otimes^n X}\otimes\psi)\circ\delta^{n+1,n+2}_{p,n+1}\circ(1_{\otimes^n X}\otimes C_\psi)\\ &\quad+\textstyle\sum_{p\in I_n}\chi^{-1}\left(\delta^{n+1,n+2}_{p,n+2}\right)\cdot(1_{\otimes^n X}\otimes\psi)\circ\delta^{n+1,n+2}_{p,n+2}\circ(1_{\otimes^n X}\otimes C_\psi)\\ &\quad+\chi^{-1}\left(\delta^{n+1,n+2}_{n+1,n+2}\right)\cdot(1_{\otimes^n X}\otimes\psi)\circ\delta^{n+1,n+2}_{n+1,n+2}\circ(1_{\otimes^n X}\otimes C_\psi)\end{aligned}$$

$$= \textstyle\sum_{p,q\in I_n:p<q}\chi^{-1}\left(\delta^{n-1,n}_{p,q}\right)\cdot(1_{\otimes^n X}\otimes\psi)\circ\tau_{(n-1,n+1)(n,n+2)}\circ(1_{\otimes^n X}\otimes C_\psi)\circ\delta^{n-1,n}_{p,q} \tag{21}$$

$$+\textstyle\sum_{p\in I_n}\chi^{-1}\left(\delta^n_p\right)\chi^{-1}\left(\tau_{(n,n+1,n+2)}\right)\cdot(1_{\otimes^n X}\otimes\psi)\circ\tau_{(n,n+1,n+2)}\circ(1_{\otimes^n X}\otimes C_\psi)\circ\delta^n_p \tag{22}$$

$$+\textstyle\sum_{p\in I_n}\chi^{-1}\left(\delta^n_p\right)\chi^{-1}\left(\tau_{(n,n+1)}\right)\cdot(1_{\otimes^n X}\otimes\psi)\circ\tau_{(n,n+1)}\circ(1_{\otimes^n X}\otimes C_\psi)\circ\delta^n_p \tag{23}$$

$$+(1_{\otimes^n X}\otimes\psi)\circ(1_{\otimes^n X}\otimes C_\psi)\,. \tag{24}$$

Making the substitution $(n-1,n,n+1,n+2)=(1,2,3,4)$, we may write $\tau_{(n-1,n+1)(n,n+2)}=1_{\otimes^{n-2}X}\otimes\tau_{(13)(24)}$, where $\tau_{(13)(24)}=\tau_{X\otimes X,X\otimes X}$ is acting on the last four factors $X\otimes X\otimes X\otimes X$ of $\otimes^{n+2}X$. Then the relation

$$(1_{\otimes^2 X}\otimes\psi)\circ\tau_{X\otimes X,X\otimes X}\circ(1_{\otimes^2 X}\otimes C_\psi)=(1_{\otimes^2 X}\otimes\psi)\circ(C_\psi\otimes 1_{\otimes^2 X})=(C_\psi\otimes\psi)=C_\psi\circ\psi$$

implies that we have

$$(1_{\otimes^n X}\otimes\psi)\circ\tau_{(n-1,n+1)(n,n+2)}\circ(1_{\otimes^n X}\otimes C_\psi)=(1_{\otimes^{n-2}X}\otimes C_\psi)\circ(1_{\otimes^{n-2}X}\otimes\psi)\,.$$

Hence it follows from (16) that we have:

$$(21)=\frac{n\,(n-1)}{2}\cdot C^{n-2}_\psi\circ\widehat{\Delta}^n_{\psi,*}. \tag{25}$$

We are now going to compute the sums (22) and (23). Making the substitution $(n,n+1,n+2)=(1,2,3)$, we may write $\tau_{(n,n+1,n+2)}=1_{\otimes^{n-1}X}\otimes\tau_{(123)}$ (resp. $\tau_{(n,n+1)}=1_{\otimes^{n-1}X}\otimes\tau_{(12)}$), where $\tau_{(123)}$ (resp. $\tau_{(12)}$) is acting on the last three factors $X\otimes X\otimes X$ of $\otimes^{n+2}X$. It follows from (18) and (19)) that we have, respectively,

$$\begin{aligned}&\chi^{-1}\left(\tau_{(n,n+1,n+2)}\right)\cdot(1_{\otimes^n X}\otimes\psi)\circ\tau_{(n,n+1,n+2)}\circ(1_{\otimes^n X}\otimes C_\psi)=1_{\otimes^n X},\\ &\chi^{-1}\left(\tau_{(n,n+1)}\right)\cdot(1_{\otimes^n X}\otimes\psi)\circ\tau_{(n,n+1)}\circ(1_{\otimes^n X}\otimes C_\psi)=1_{\otimes^n X}.\end{aligned}$$

Hence we find

$$(22)=(23)=\textstyle\sum_{p\in I_n}\chi^{-1}\left(\delta^n_p\right)\delta^n_p=n\cdot e^\chi_{R_{S_{n-1}\backslash S_n}}. \tag{26}$$

Finally, it follows from (13) that we have

$$(24)=\nu_* r_X\cdot 1_{\otimes^n X}. \tag{27}$$

It now follows from (25), (26) and (27) that we have, as claimed,

$$\begin{aligned}\frac{(n+2)(n+1)}{2}\cdot\widehat{\Delta}^{n+2}_{\psi,*}\circ C^{n}_{\psi}&=(21)+(22)+(23)+(24)\\&=\frac{n(n-1)}{2}\cdot C^{n-2}_{\psi}\circ\widehat{\Delta}^{n}_{\psi,*}+2n\cdot e^{\chi}_{R_{S_{n-1}\setminus S_n}}+\nu_* r_X\cdot 1_{\otimes^n X}.\end{aligned}$$

□

The following definition will be useful in the following subsections.

Definition 3.5. *We say the a morphism $f:M\to M$ is diagonalizable if there is an isomorphism $M\simeq\bigoplus_{\lambda\in End(\mathbb{I})}M_{f,\lambda}$ such that $M_{f,\lambda}=0$ for almost every λ and $f\simeq\bigoplus_{\lambda\in End(\mathbb{I})}f_\lambda$ via this isomorphism, with $f_\lambda=\lambda:M_{f,\lambda}\to M_{f,\lambda}$ the multiplication by $\lambda\in End(\mathbb{I})$. In this case, we call the set*

$$\sigma(f):=\{\lambda:M_{f,\lambda}\neq 0\}\subset End(\mathbb{I})$$

the spectrum of f.

It will be also convenient to introduce the following definition.

Definition 3.6. *If $S\subset End(\mathbb{I})$ we say that S is strictly positive (resp. positive, strictly negative or negative) and we write $S>0$ (resp. $S\geq 0$, $S<0$ or $S\leq 0$) to mean that there exists an ordered field $(K,\geq)$ such that $S\subset K\subset End(\mathbb{I})$ and $s>0$ (resp. $s\geq 0$, $s<0$ or $s\leq 0$) in K for every $s\in S$. If $s\in End(\mathbb{I})$, we write $s>0$ (resp. $S\geq 0$, $S<0$ or $S\leq 0$) to mean that $S>0$ (resp. $S\geq 0$, $S<0$ or $S\leq 0$) with $S=\{s\}$.*

3.1. **Laplace operators attached to $\mathbb{I}$-valued perfect alternating pairings.** We suppose in this subsection that we are given $\psi:X\otimes X\to\mathbb{I}$ which is perfect, i.e. such that the associated hom valued morphism is an isomorphism, and alternating, i.e. $\psi\circ\tau_{X,X}=-\psi$. It follows from Lemma 3.1 that we have $\Delta^{n}_{\psi,s}=0$ and, hence, we concentrate on $\Delta^{n}_{\psi,a}$. We set $r_X:=\mathrm{rank}(X)$ in the subsequent discussion.

Proposition 3.7. *When $r_X<0$ we have that $\Delta^{n+2}_{\psi,a}\circ C^{n}_{\psi,a}$ when $n\geq 0$ (resp. $C^{n-2}_{\psi,a}\circ\Delta^{n}_{\psi,a}$ when $n\geq 2$) is diagonalizable, with spectrum*

$$\sigma\left(\Delta^{n+2}_{\psi,a}\circ C^{n}_{\psi,a}\right)>0\ \textit{(resp. }\sigma\left(C^{n-2}_{\psi,a}\circ\Delta^{n}_{\psi,a}\right)\geq 0\textit{).}$$

Proof. It will be convenient to set $\delta^{n}_{\psi,a}:=\frac{n(n-1)}{2}\cdot\Delta^{n}_{\psi,a}$, so that Lemma 3.4 gives $\delta^{2}_{\psi,a}\circ C^{0}_{\psi,a}=-r_X$, $\delta^{3}_{\psi,a}\circ C^{1}_{\psi,a}=(2-r_X)\cdot 1_X$ and, for every $n\geq 2$,

$$\delta^{n+2}_{\psi,a}\circ C^{n}_{\psi,a}-C^{n-2}_{\psi,a}\circ\delta^{n}_{\psi,a}=(2n-r_X)\cdot 1_{\wedge^n X}. \tag{28}$$

In particular, we see that $\delta^{n+2}_{\psi,a}\circ C^{n}_{\psi,a}$ is diagonalizable for $n=0,1$ with $\sigma\left(\delta^{2}_{\psi,a}\circ C^{0}_{\psi,a}\right)=\{-r_X\}>0$ and $\sigma\left(\delta^{3}_{\psi,a}\circ C^{1}_{\psi,a}\right)=\{2-r_X\}>0$. We can now assume that $n\geq 2$ and that, by induction, $\delta^{n}_{\psi,a}\circ C^{n-2}_{\psi,a}$ is diagonalizable with spectrum $\sigma\left(\delta^{n}_{\psi,a}\circ C^{n-2}_{\psi,a}\right)>0$ and we claim that this implies both that $\Delta^{n+2}_{\psi,a}\circ C^{n}_{\psi,a}$ is diagonalizable with spectrum >0 and that $C^{n-2}_{\psi,a}\circ\Delta^{n}_{\psi,a}$ is diagonalizable with spectrum ≥ 0. Here and in the following, the ordered field $(K,\geq)$ in the definition of being positive is always taken to be the one appearing in the definition of $-r_X>0$.

Since $\delta^{n}_{\psi,a}\circ C^{n-2}_{\psi,a}$ is diagonalizable with spectrum >0, we have that $\delta^{n}_{\psi,a}\circ C^{n-2}_{\psi,a}$ is an isomorphism. It now follows from an abstract non-sense that there is a biproduct decomposition

$$\wedge^n X\simeq\ker\left(C^{n-2}_{\psi,a}\circ\delta^{n}_{\psi,a}\right)\oplus\wedge^{n-2}X$$

such that

$$C^{n-2}_{\psi,a}\circ\delta^{n}_{\psi,a}\simeq 0\oplus\left(\delta^{n}_{\psi,a}\circ C^{n-2}_{\psi,a}\right). \tag{29}$$

Since $\delta^n_{\psi,a} \circ C^{n-2}_{\psi,a}$ is diagonalizable with spectrum $\sigma\left(\delta^n_{\psi,a} \circ C^{n-2}_{\psi,a}\right) > 0$, it follows from (29) that $\sigma\left(C^{n-2}_{\psi,a} \circ \delta^n_{\psi,a}\right)$ is diagonalizable with spectrum

$$\sigma\left(C^{n-2}_{\psi,a} \circ \delta^n_{\psi,a}\right) \subset \{0\} \cup \sigma\left(\delta^n_{\psi,a} \circ C^{n-2}_{\psi,a}\right) \geq 0.$$

It now follows from (28) that $\delta^{n+2}_{\psi,a} \circ C^n_{\psi,a}$ is diagonalizable with spectrum

$$\sigma\left(\Delta^{n+2}_{\psi,a} \circ C^n_{\psi,a}\right) \subset \left\{\lambda + (2n - r_X) : \lambda \in \sigma\left(C^{n-2}_{\psi,a} \circ \delta^n_{\psi,a}\right)\right\} > 0.$$

□

Corollary 3.8. *When $r_X < 0$ we have that, for every $n \geq 2$, the Laplace operator $\Delta^n_{\psi,a}$ has a section $s^{n-2}_{\psi,a} : \wedge^{n-2} X \to \wedge^n X$ such that $\Delta^n_{\psi,a} \circ s^{n-2}_{\psi,a} = 1_{\wedge^{n-2} X}$ and, in particular,* $\ker\left(\Delta^n_{\psi,a}\right)$ *exists.*

Proof. Indeed $\Delta^n_{\psi,a} \circ C^{n-2}_{\psi,a}$ is diagonalizable with spectrum $\sigma\left(\Delta^n_{\psi,a} \circ C^{n-2}_{\psi,a}\right) > 0$ by Proposition 3.7 and, in particular, it is an isomorphism. □

3.2. **Laplace operators attached to $\mathbb{I}$-valued perfect symmetric pairings.** We suppose in this subsection that we are given $\psi : X \otimes X \to \mathbb{I}$ which is perfect, i.e. such that the associated hom valued morphism is an isomorphism, and symmetric, i.e. $\psi \circ \tau_{X,X} = \psi$. It follows from Lemma 3.1 that we have $\Delta^n_{\psi,a} = 0$ and, hence, we concentrate on $\Delta^n_{\psi,s}$. We set $r_X := \operatorname{rank}(X)$ in the subsequent discussion.

Proposition 3.9. *When $r_X > 0$ we have that $\Delta^{n+2}_{\psi,s} \circ C^n_{\psi,s}$ when $n \geq 0$ (resp. $C^{n-2}_{\psi,s} \circ \Delta^n_{\psi,s}$ when $n \geq 2$) is diagonalizable, with spectrum*

$$\sigma\left(\Delta^{n+2}_{\psi,a} \circ C^n_{\psi,a}\right) > 0 \text{ (resp. } \sigma\left(C^{n-2}_{\psi,a} \circ \Delta^n_{\psi,a}\right) \geq 0\text{)}.$$

Proof. Setting $\delta^n_{\psi,s} := \frac{n(n-1)}{2} \cdot \Delta^n_{\psi,s}$, Lemma 3.4 gives the equalities $\delta^2_{\psi,s} \circ C^0_{\psi,s} = r_X$, $\delta^3_{\psi,s} \circ C^1_{\psi,s} = (2 + r_X) \cdot 1_X$ and, for every $n \geq 2$,

$$\delta^{n+2}_{\psi,s} \circ C^n_{\psi,s} - C^{n-2}_{\psi,s} \circ \delta^n_{\psi,s} = (2n + r_X) \cdot 1_{\vee^n X}.$$

Then the proof is just a copy of those of Proposition 3.7. □

The following corollary may be deduced from Proposition 3.9 in the same way as Corollary 3.8 was deduced from Proposition 3.7.

Corollary 3.10. *When $r_X > 0$ we have that, for every $n \geq 2$, the Laplace operator $\Delta^n_{\psi,s}$ has a section $s^{n-2}_{\psi,s} : \vee^{n-2} X \to \vee^n X$ such that $\Delta^n_{\psi,s} \circ s^{n-2}_{\psi,s} = 1_{\vee^{n-2} X}$ and, in particular,* $\ker\left(\Delta^n_{\psi,s}\right)$ *exists.*

3.3. **Laplace operators attached to perfect pairings valued in squares of invertible objects.** We suppose in this subsection that we are given a perfect pairing $\psi : X \otimes X \to Z$, i.e. such that $f_\psi : X \to \hom(X, Z)$ is an isomorphism, and that we have $Z \simeq \mathbb{L}^{\otimes 2}$, where $\mathbb{L}$ an invertible object. We assume that ψ is alternating or symmetric, i.e. $\psi \circ \tau_{X,X} = \chi(\tau_{X,X}) \cdot \psi$, where $\chi \in \{\varepsilon, 1\}$. As above, we define $* := a$ when $\chi = \varepsilon$ and $* := s$ when $\chi = 1$ and we write $*^k X := \wedge^k X$ when $\chi = \varepsilon$ and $*^k X := \vee^k X$ when $\chi = 1$. It follows from Lemma 3.1 that we have $\Delta^n_{\psi,*} = 0$ if $\{*\} = \{a, s\} - \{*\}$ and , hence, we concentrate on $\Delta^n_{\psi,*}$. We set $r_X := \operatorname{rank}(X)$ and $r_{\mathbb{L}} := \operatorname{rank}(\mathbb{L})$ in the subsequent discussion, so that $r_{\mathbb{L}} \in \{\pm 1\}$.

Let $\tau_{\delta_k} : \otimes^k (X \otimes \mathbb{L}) \overset{\sim}{\to} \left(\otimes^k X\right) \otimes \mathbb{L}^{\otimes k}$ be the isomorphism induced by the permutation $\delta_k \in S_{2k}$ such that $\delta_k(2i-1) = i$ and $\delta_k(2i) = k + i$ for every $i \in I_k$. It is not difficult to show, using [De3, 7.2 Lemme], that one has

$$e^k_{X \otimes \mathbb{L},a} \simeq e^k_{X,a} \otimes 1_{\mathbb{L}^{\otimes k}} \text{ and } e^k_{X \otimes \mathbb{L},s} \simeq e^k_{X,s} \otimes 1_{\mathbb{L}^{\otimes k}} \text{ if } r_{\mathbb{L}} = 1, \tag{30}$$

$$e^k_{X \otimes \mathbb{L},s} \simeq e^k_{X,a} \otimes 1_{\mathbb{L}^{\otimes k}} \text{ and } e^k_{X \otimes \mathbb{L},a} \simeq e^k_{X,s} \otimes 1_{\mathbb{L}^{\otimes k}} \text{ if } r_{\mathbb{L}} = -1. \tag{31}$$

Lemma 3.11. *Suppose that $\varphi : X \otimes X \to \mathbb{L}^{\otimes 2}$ is alternating (resp. symmetric) and consider the composite*

$$\varphi_{\mathbb{L}^{-1}} : \left(X \otimes \mathbb{L}^{-1}\right) \otimes \left(X \otimes \mathbb{L}^{-1}\right) \overset{1_X \otimes \tau_{\mathbb{L}^{-1},X} \otimes 1_{\mathbb{L}^{-1}}}{\to} X \otimes X \otimes \mathbb{L}^{\otimes -2} \overset{\varphi \otimes 1_{\mathbb{L}^{\otimes -2}}}{\to} \mathbb{L}^{\otimes 2} \otimes \mathbb{L}^{\otimes -2} \overset{ev_{\mathbb{L}^{\otimes -2}}}{\to} \mathbb{I}.$$

(*a*) *If $r_{\mathbb{L}} = 1$, the morphism $\varphi_{\mathbb{L}^{-1}}$ is alternating (resp. symmetric) and the following diagrams are commutative*

$$\begin{tikzcd}
\wedge^n \left(X \otimes \mathbb{L}^{-1}\right) \arrow[r, "\Delta^n_{\varphi_{\mathbb{L}^{-1}},a}"] \arrow[d, "\tau_{\delta_n}"] & \wedge^{n-2} \left(X \otimes \mathbb{L}^{-1}\right) \arrow[d, "\left(1_{\wedge^{n-2}X} \otimes \tau_{\mathbb{L}^{\otimes -(n-2)},\mathbb{L}^{\otimes 2}} \otimes 1_{\mathbb{L}^{\otimes -2}}\right) \circ \tau_{\delta_{n-2}}"] \\
\left(\wedge^n X\right) \otimes \mathbb{L}^{\otimes -n} \arrow[r, "\Delta^n_{\varphi,a} \otimes 1_{\mathbb{L}^{\otimes -n}}"] & \left(\wedge^{n-2} X\right) \otimes \mathbb{L}^{\otimes 2} \otimes \mathbb{L}^{\otimes -n},
\end{tikzcd}
\qquad
\begin{tikzcd}
\vee^n \left(X \otimes \mathbb{L}^{-1}\right) \arrow[r, "\Delta^n_{\varphi_{\mathbb{L}^{-1}},s}"] \arrow[d, "\tau_{\delta_n}"] & \vee^{n-2} \left(X \otimes \mathbb{L}^{-1}\right) \arrow[d, "\left(1_{\vee^{n-2}X} \otimes \tau_{\mathbb{L}^{\otimes -(n-2)},\mathbb{L}^{\otimes 2}} \otimes 1_{\mathbb{L}^{\otimes -2}}\right) \circ \tau_{\delta_{n-2}}"] \\
\left(\vee^n X\right) \otimes \mathbb{L}^{\otimes -n} \arrow[r, "\Delta^n_{\varphi,s} \otimes 1_{\mathbb{L}^{\otimes -n}}"] & \left(\vee^{n-2} X\right) \otimes \mathbb{L}^{\otimes 2} \otimes \mathbb{L}^{\otimes -n},
\end{tikzcd}$$

where $\tau_{\delta_k} : \wedge^k (X \otimes \mathbb{L}) \overset{\sim}{\to} \left(\wedge^k X\right) \otimes \mathbb{L}^{\otimes k}$ and $\tau_{\delta_k} : \vee^k (X \otimes \mathbb{L}) \overset{\sim}{\to} \left(\vee^k X\right) \otimes \mathbb{L}^{\otimes k}$ are the isomorphisms induced by (*30*).

(*b*) *If $r_{\mathbb{L}} = -1$ the morphism $\varphi_{\mathbb{L}^{-1}}$ is symmetric (resp. alternating) and the following diagrams are commutative:*

$$\begin{tikzcd}
\vee^n \left(X \otimes \mathbb{L}^{-1}\right) \arrow[r, "\Delta^n_{\varphi_{\mathbb{L}^{-1}},s}"] \arrow[d, "\tau_{\delta_n}"] & \vee^{n-2} \left(X \otimes \mathbb{L}^{-1}\right) \arrow[d, "\left(1_{\wedge^{n-2}X} \otimes \tau_{\mathbb{L}^{\otimes -(n-2)},\mathbb{L}^{\otimes 2}} \otimes 1_{\mathbb{L}^{\otimes -2}}\right) \circ \tau_{\delta_{n-2}}"] \\
\left(\wedge^n X\right) \otimes \mathbb{L}^{\otimes -n} \arrow[r, "\Delta^n_{\varphi,a} \otimes 1_{\mathbb{L}^{\otimes -n}}"] & \left(\wedge^{n-2} X\right) \otimes \mathbb{L}^{\otimes 2} \otimes \mathbb{L}^{\otimes -n},
\end{tikzcd}
\qquad
\begin{tikzcd}
\wedge^n \left(X \otimes \mathbb{L}^{-1}\right) \arrow[r, "\Delta^n_{\varphi_{\mathbb{L}^{-1}},a}"] \arrow[d, "\tau_{\delta_n}"] & \wedge^{n-2} \left(X \otimes \mathbb{L}^{-1}\right) \arrow[d, "\left(1_{\vee^{n-2}X} \otimes \tau_{\mathbb{L}^{\otimes -(n-2)},\mathbb{L}^{\otimes 2}} \otimes 1_{\mathbb{L}^{\otimes -2}}\right) \circ \tau_{\delta_{n-2}}"] \\
\left(\vee^n X\right) \otimes \mathbb{L}^{\otimes -n} \arrow[r, "\Delta^n_{\varphi,s} \otimes 1_{\mathbb{L}^{\otimes -n}}"] & \left(\vee^{n-2} X\right) \otimes \mathbb{L}^{\otimes 2} \otimes \mathbb{L}^{\otimes -n},
\end{tikzcd}$$

where $\tau_{\delta_k} : \vee^k (X \otimes \mathbb{L}) \overset{\sim}{\to} \left(\wedge^k X\right) \otimes \mathbb{L}^{\otimes k}$ and $\tau_{\delta_k} : \wedge^k (X \otimes \mathbb{L}) \overset{\sim}{\to} \left(\vee^k X\right) \otimes \mathbb{L}^{\otimes k}$ are the isomorphisms induced by (*31*).

(*c*) *Writing $f_\varphi : X \to \hom\left(X, \mathbb{L}^{\otimes 2}\right)$ and $f_{\varphi_{\mathbb{L}^{-1}}} : X \otimes \mathbb{L}^{-1} \to \left(X \otimes \mathbb{L}^{-1}\right)^\vee$ for the associated morphisms we have that f_φ is an isomorphism is and only if $f_{\varphi_{\mathbb{L}^{-1}}}$ is an isomorphism.*

Proof. (*a*-*b*) We first claim that the following diagram is commutative:

$$\begin{tikzcd}
\otimes^n \left(X \otimes \mathbb{L}^{-1}\right) \arrow[r, "\Delta^n_{\varphi_{\mathbb{L}^{-1}}}"] \arrow[d, "\tau_{\delta_n}"] & \otimes^{n-2} \left(X \otimes \mathbb{L}^{-1}\right) \arrow[dr, bend left, "\tau_{\delta_{n-2}}"] & \\
\left(\otimes^n X\right) \otimes \mathbb{L}^{\otimes -n} \arrow[r, "\Delta^n_\varphi \otimes 1_{\mathbb{L}^{\otimes -n}}"'] & \left(\otimes^{n-2} X\right) \otimes \mathbb{L}^{\otimes 2} \otimes \mathbb{L}^{\otimes -n} \arrow[r, "1_{\otimes^{n-2}X} \otimes \tau_{\mathbb{L}^{\otimes 2},\mathbb{L}^{\otimes -(n-2)}} \otimes 1_{\mathbb{L}^{\otimes -2}}"'] & \left(\otimes^{n-2} X\right) \otimes \mathbb{L}^{\otimes -(n-2)}
\end{tikzcd} \tag{32}$$

A tedious computation reveals that:

$$\left(\tau_{\delta_{n-2}} \otimes 1_X \otimes \tau_{\mathbb{L}^{-1},X} \otimes 1_{\mathbb{L}^{-1}}\right) = \left(1_{\otimes^{n-2}X} \otimes \tau_{\otimes^2 X, \mathbb{L}^{\otimes -(n-2)}} \otimes 1_{\mathbb{L}^{\otimes -2}}\right) \circ \tau_{\delta_n} \tag{33}$$

Hence we have:

$$\begin{aligned}
\tau_{\delta_{n-2}} \circ \Delta^{n-2}_{\varphi_{\mathbb{L}^{-1}}} &= \tau_{\delta_{n-2}} \circ \left(1_{\otimes^{n-2}(X \otimes \mathbb{L}^{-1})} \otimes \varphi_{\mathbb{L}^{-1}}\right) = \tau_{\delta_{n-2}} \circ \left(1_{\otimes^{n-2}(X \otimes \mathbb{L}^{-1})} \otimes \varphi \otimes 1_{\mathbb{L}^{\otimes -2}}\right) \\
&\quad \circ \left(1_{\otimes^{n-2}(X \otimes \mathbb{L}^{-1})} \otimes 1_X \otimes \tau_{\mathbb{L}^{-1},X} \otimes 1_{\mathbb{L}^{-1}}\right) \\
&= \left(1_{(\otimes^{n-2}X) \otimes \mathbb{L}^{\otimes -(n-2)}} \otimes \varphi \otimes 1_{\mathbb{L}^{\otimes -2}}\right) \circ \left(\tau_{\delta_{n-2}} \otimes 1_X \otimes \tau_{\mathbb{L}^{-1},X} \otimes 1_{\mathbb{L}^{-1}}\right) \text{ (by (33))} \\
&= \left(1_{(\otimes^{n-2}X) \otimes \mathbb{L}^{\otimes -(n-2)}} \otimes \varphi \otimes 1_{\mathbb{L}^{\otimes -2}}\right) \circ \left(1_{\otimes^{n-2}X} \otimes \tau_{\otimes^2 X, \mathbb{L}^{\otimes -(n-2)}} \otimes 1_{\mathbb{L}^{\otimes -2}}\right) \circ \tau_{\delta_n} \\
&= \left(1_{\otimes^{n-2}X} \otimes \tau_{\mathbb{L}^{\otimes 2}, \mathbb{L}^{\otimes -(n-2)}} \otimes 1_{\mathbb{L}^{\otimes -2}}\right) \circ \left(1_{\otimes^{n-2}X} \otimes \varphi \otimes 1_{\mathbb{L}^{\otimes -n}}\right) \circ \tau_{\delta_n} \\
&= \left(1_{\otimes^{n-2}X} \otimes \tau_{\mathbb{L}^{\otimes 2}, \mathbb{L}^{\otimes -(n-2)}} \otimes 1_{\mathbb{L}^{\otimes -2}}\right) \circ \left(\Delta^n_\varphi \otimes 1_{\mathbb{L}^{\otimes -n}}\right) \circ \tau_{\delta_n},
\end{aligned}$$

showing that (32) is commutative. The claimed commutative diagrams in (*a*) and (*b*) now follows from (30), (31) and the commutativity of (32).

We view $1_X \otimes \tau_{\mathbb{L}^{-1},X} \otimes 1_{\mathbb{L}^{-1}} = \tau_{(23)}$ and $\tau_{X \otimes \mathbb{L}^{-1}, X \otimes \mathbb{L}^{-1}} = \tau_{(13)(24)}$ as induced by permutations in S_4 and then, noticing that $(23)(13)(24) = (1243) = (12)(34)(23)$ and that we have $\varphi \circ \tau_{X,X} = \chi(\tau_{X,X}) \cdot \psi$ with

$\chi = \varepsilon$ (resp. $\chi = 1$), we find

$$\begin{aligned}
\varphi_{\mathbb{L}^{-1}} \circ \tau_{X\otimes\mathbb{L}^{-1},X\otimes\mathbb{L}^{-1}} &= ev_{\mathbb{L}^{\otimes-2}} \circ (\varphi \otimes 1_{\mathbb{L}^{\otimes-2}}) \circ \left(1_X \otimes \tau_{\mathbb{L}^{-1},X} \otimes 1_{\mathbb{L}^{-1}}\right) \circ \tau_{X\otimes\mathbb{L}^{-1},X\otimes\mathbb{L}^{-1}} \\
&= ev_{\mathbb{L}^{\otimes-2}} \circ (\varphi \otimes 1_{\mathbb{L}^{\otimes-2}}) \circ \tau_{(23)} \circ \tau_{(13)(24)} = e \circ (\varphi \otimes 1_{\mathbb{L}^{\otimes-2}}) \circ \tau_{(1243)} \\
&= ev_{\mathbb{L}^{\otimes-2}} \circ (\varphi \otimes 1_{\mathbb{L}^{\otimes-2}}) \circ \tau_{(12)} \circ \tau_{(34)} \circ \tau_{(23)} \\
&= ev_{\mathbb{L}^{\otimes-2}} \circ (\varphi \otimes 1_{\mathbb{L}^{\otimes-2}}) \circ \left(\tau_{X,X} \otimes \tau_{\mathbb{L}^{-1},\mathbb{L}^{-1}}\right) \circ \left(1_X \otimes \tau_{\mathbb{L}^{-1},X} \otimes 1_{\mathbb{L}^{-1}}\right) \\
&= \chi\left(\tau_{X,X}\right) \cdot ev_{\mathbb{L}^{\otimes-2}} \circ \left(\varphi \otimes \tau_{\mathbb{L}^{-1},\mathbb{L}^{-1}}\right) \circ \left(1_X \otimes \tau_{\mathbb{L}^{-1},X} \otimes 1_{\mathbb{L}^{-1}}\right).
\end{aligned}$$

It follows from [De3, 7.2 Lemme] that we have $\tau_{\mathbb{L}^{-1},\mathbb{L}^{-1}} = r_{\mathbb{L}}$, so that we find

$$\varphi_{\mathbb{L}^{-1}} \circ \tau_{X\otimes\mathbb{L}^{-1},X\otimes\mathbb{L}^{-1}} = r_{\mathbb{L}}\chi\left(\tau_{X,X}\right) \cdot \varphi_{\mathbb{L}^{-1}}.$$

(c) This is left to the reader. □

Proposition 3.12. *When $\chi\left(\tau_{X,X}\right) r_X > 0$ we have that, for every $n \geq 2$, the Laplace operator*

$$\Delta^n_{\psi,*} : *^n X \to *^{n-2} X$$

has a section $s^{n-2}_{\psi,*} : *^{n-2} X \to *^n X$ such that $\Delta^n_{\psi,*} \circ s^{n-2}_{\psi,*} = 1_{*^{n-2}X}$ and, in particular, $\ker\left(\Delta^n_{\psi,*}\right)$ *exists.*

Proof. If $\sigma : Z \xrightarrow{\sim} \mathbb{L}^{\otimes 2}$ is our given isomorphism, we have that $\Delta^n_{\sigma\circ\psi,*} = (1_{\wedge^{n-2}X} \otimes \sigma) \circ \Delta^n_{\psi,*}$ has a section if and only if $\Delta^n_{\psi,a}$ has a section: hence we may assume that $Z = \mathbb{L}^{\otimes 2}$. We can now consider the composite:

$$\psi_{\mathbb{L}^{-1}} : \left(X \otimes \mathbb{L}^{-1}\right) \otimes \left(X \otimes \mathbb{L}^{-1}\right) \overset{1_X\otimes\tau_{\mathbb{L}^{-1},X}\otimes 1_{\mathbb{L}^{-1}}}{\to} X \otimes X \otimes \mathbb{L}^{\otimes-2} \overset{\psi\otimes 1_{\mathbb{L}^{\otimes-2}}}{\to} \mathbb{L}^{\otimes 2} \otimes \mathbb{L}^{\otimes-2} \overset{ev_{\mathbb{L}^{\otimes-2}}}{\to} \mathbb{I}.$$

When $r_{\mathbb{L}} = 1$ (resp. $r_{\mathbb{L}} = -1$), Lemma 3.11 (a) (resp. (b)) shows that $\Delta^n_{\psi,*}$ has a section if and only if $\Delta^n_{\varphi_{\mathbb{L}^{-1}},*}$ (resp. $\Delta^n_{\varphi_{\mathbb{L}^{-1}},*}$) has a section and that $\varphi_{\mathbb{L}^{-1}}$ satisfies $\varphi_{\mathbb{L}^{-1}} \circ \tau_{X\otimes\mathbb{L}^{-1},X\otimes\mathbb{L}^{-1}} = \chi\left(\tau_{X,X}\right) \cdot \varphi_{\mathbb{L}^{-1}}$ (resp. $\varphi_{\mathbb{L}^{-1}} \circ \tau_{X\otimes\mathbb{L}^{-1},X\otimes\mathbb{L}^{-1}} = -\chi\left(\tau_{X,X}\right) \cdot \varphi_{\mathbb{L}^{-1}}$). It follows from this last relation that, if we define $\varepsilon_{X\otimes\mathbb{L}^{-1},X\otimes\mathbb{L}^{-1}}$ by the rule $\varphi_{\mathbb{L}^{-1}} \circ \tau_{X\otimes\mathbb{L}^{-1},X\otimes\mathbb{L}^{-1}} = \varepsilon_{X\otimes\mathbb{L}^{-1},X\otimes\mathbb{L}^{-1}} \cdot \varphi_{\mathbb{L}^{-1}}$, then we have $\varepsilon_{X\otimes\mathbb{L}^{-1},X\otimes\mathbb{L}^{-1}} = \chi\left(\tau_{X,X}\right)$ (resp. $\varepsilon_{X\otimes\mathbb{L}^{-1},X\otimes\mathbb{L}^{-1}} = -\chi\left(\tau_{X,X}\right)$) and

$$\varepsilon_{X\otimes\mathbb{L}^{-1},X\otimes\mathbb{L}^{-1}} r_{X\otimes\mathbb{L}^{-1}} = \varepsilon_{X\otimes\mathbb{L}^{-1},X\otimes\mathbb{L}^{-1}} r_X r_{\mathbb{L}} = \chi\left(\tau_{X,X}\right) r_X > 0.$$

It follows that we may apply to $\psi_{\mathbb{L}^{-1}}$ Corollary 3.8, when $\varepsilon_{X\otimes\mathbb{L}^{-1},X\otimes\mathbb{L}^{-1}} = -1$, or Corollary 3.10, when $\varepsilon_{X\otimes\mathbb{L}^{-1},X\otimes\mathbb{L}^{-1}} = 1$, to deduce that $\Delta^n_{\varphi_{\mathbb{L}^{-1}},*}$ has a section. □

4. Laplace and Dirac operators for the alternating algebras

In this section we assume that we are given an object $V \in \mathcal{C}$such that $\wedge^g V$ is invertible. If X is an object we set $r_X := \operatorname{rank}(X)$, so that $r_{\wedge^g V} \in \{\pm 1\}$, and we use the shorthand $r := r_V$.

4.1. Preliminary lemmas.

We define

$$\psi^V_{i,1} : \wedge^i V \otimes V \overset{\varphi_{i,1}}{\to} \wedge^{i+1} V \overset{D^{i+1,g}}{\to} \wedge^{g-i-1} V^\vee \otimes \wedge^g V^{\vee\vee},$$

and

$$\begin{aligned}
\overline{\psi}^V_{g-i,g-i-1} \quad : \quad & \wedge^{g-i} V \otimes \wedge^{g-i-1} V^\vee \overset{D^{g-i,g}\otimes 1_{\wedge^{g-i-1}V^\vee}}{\to} \wedge^i V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^{g-i-1} V^\vee \overset{\varphi^{13}_{i,g-i-1}}{\to} \wedge^{g-1} V^\vee \otimes \wedge^g V^{\vee\vee} \\
& \overset{D_{g-1,g}\otimes 1_{\wedge^g V^{\vee\vee}}}{\to} V \otimes \wedge^g V^\vee \otimes \wedge^g V^{\vee\vee} \overset{1_V\otimes ev^{g,\tau}_{V^\vee,a}}{\to} V.
\end{aligned}$$

We may also consider

$$\psi^V_{g-i,1} : \wedge^{g-i} V \otimes V \overset{\varphi_{g-i,1}}{\to} \wedge^{g-i+1} V \overset{D^{g-i+1,g}}{\to} \wedge^{i-1} V^\vee \otimes \wedge^g V^{\vee\vee},$$

and

$$\begin{aligned}
\overline{\psi}^V_{i,i-1} \quad : \quad & \wedge^i V \otimes \wedge^{i-1} V^\vee \overset{D^{i,g}\otimes 1_{\wedge^{i-1}V^\vee}}{\to} \wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^{i-1} V^\vee \overset{\varphi^{13}_{g-i,i-1}}{\to} \wedge^{g-1} V^\vee \otimes \wedge^g V^{\vee\vee} \\
& \overset{D_{g-1,g}\otimes 1_{\wedge^g V^{\vee\vee}}}{\to} V \otimes \wedge^g V^\vee \otimes \wedge^g V^{\vee\vee} \overset{1_V\otimes ev^{g,\tau}_{V^\vee,a}}{\to} V.
\end{aligned}$$

Lemma 4.1. *Setting*

$$\rho_V^{i,g-i} := (-1)^{(g-1)}\, r_{\wedge^g V} \binom{g}{g-1}^{-1} \binom{g}{g-i}^{-1} \binom{r-1}{g-1} \binom{r-i}{g-i} g,$$
$$\nu_V^{g-i,1} := (-1)^{g-i}\, i \text{ and } \nu_V^{i,1} := (-1)^{i(g-i-1)}\, (g-i)$$

the following diagram is commutative:

$$\begin{tikzcd}
\wedge^i V \otimes \wedge^{g-i} V \otimes V \arrow[r, "{\left(1_{\wedge^i V} \otimes \psi_{g-i,1}, \left(1_{\wedge^{g-i} V} \otimes \psi_{i,1}\right) \circ \left(\tau_{\wedge^i V, \wedge^{g-i} V} \otimes 1_V\right)\right)}"] \arrow[d, "\varphi_{i,g-i} \otimes 1_V"'] & \wedge^i V \otimes \wedge^{i-1} V^\vee \otimes \wedge^g V^{\vee\vee} \oplus \wedge^{g-i} V \otimes \wedge^{g-i-1} V^\vee \otimes \wedge^g V^{\vee\vee} \arrow[d, "\nu_V^{g-i,1} \cdot \overline{\psi}_{i,i-1} \otimes 1_{\wedge^g V^{\vee\vee}} \oplus \nu_V^{i,1} \cdot \overline{\psi}_{g-i,g-i-1} \otimes 1_{\wedge^g V^{\vee\vee}}"] \\
\wedge^g V \otimes V \arrow[r, "\rho_V^{i,g-i} \cdot \tau_{\wedge^g V^{\vee\vee}, V} \circ (i_{\wedge^g V} \otimes 1_V)"] & V \otimes \wedge^g V^{\vee\vee}.
\end{tikzcd}$$

Proof. Set

$$\begin{aligned}
a &:= (-1)^{g-i}\, i \cdot \left(\varphi^{13}_{g-i,i-1} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(D^{i,g} \otimes 1_{\wedge^{i-1} V^\vee \otimes \wedge^g V^{\vee\vee}}\right) \circ \left(1_{\wedge^i V} \otimes D^{g-i+1,g}\right) \circ \left(1_{\wedge^i V} \otimes \varphi_{g-i,1}\right) \\
&= (-1)^{g-i}\, i \cdot \left(\varphi^{13}_{g-i,i-1} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(D^{i,g} \otimes D^{g-i+1,g}\right) \circ \left(1_{\wedge^i V} \otimes \varphi_{g-i,1}\right), \\
b &:= (-1)^{i(g-i-1)}\, (g-i) \cdot \left(\varphi^{13}_{i,g-i-1} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(D^{g-i,g} \otimes 1_{\wedge^{g-i-1} V^\vee \otimes \wedge^g V^{\vee\vee}}\right) \circ \left(1_{\wedge^{g-i} V} \otimes D^{i+1,g}\right) \\
&\quad \circ \left(1_{\wedge^{g-i} V} \otimes \varphi_{i,1}\right) \circ \left(\tau_{\wedge^i V, \wedge^{g-i} V} \otimes 1_V\right) \\
&= (-1)^{i(g-i-1)}\, (g-i) \cdot \left(\varphi^{13}_{i,g-i-1} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(D^{g-i,g} \otimes D^{i+1,g}\right) \\
&\quad \circ \left(1_{\wedge^{g-i} V} \otimes \varphi_{i,1}\right) \circ \left(\tau_{\wedge^i V, \wedge^{g-i} V} \otimes 1_V\right), \\
\phi &:= \left(1_V \otimes ev^{g,\tau}_{V^\vee,a} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(D_{g-1,g} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}\right).
\end{aligned}$$

Then we have $\left(\overline{\psi}^V_{i,i-1} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(1_{\wedge^i V} \otimes \psi^V_{g-i,1}\right) = \phi \circ a$ and $\left(\overline{\psi}^V_{g-i,g-i-1} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(1_{\wedge^{g-i} V} \otimes \psi^V_{i,1}\right) \circ \left(\tau_{\wedge^i V, \wedge^{g-i} V} \otimes 1_V\right) = \phi \circ b$. With these notations, it follows from Proposition 2.3 that we have, setting $\rho := r_{\wedge^g V}\, g \binom{g}{g-i}^{-1} \binom{r-i}{g-i}$[3],

$$a + b = \rho \cdot \left(1_{\wedge^{g-1} V^\vee \otimes \wedge^g V^{\vee\vee}} \otimes i_{\wedge^g V}\right) \circ \left(D^{1,g} \otimes \varphi_{i,g-i}\right) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V}. \tag{34}$$

Hence we find

$$\begin{aligned}
&\left(\overline{\psi}^V_{i,i-1} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(1_{\wedge^i V} \otimes \psi^V_{g-i,1}\right) + \left(\overline{\psi}^V_{g-i,g-i-1} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(1_{\wedge^{g-i} V} \otimes \psi^V_{i,1}\right) \circ \left(\tau_{\wedge^i V, \wedge^{g-i} V} \otimes 1_V\right) \\
&= \phi \circ (a+b) \ \text{(by (34))} = \rho \cdot \phi \circ \left(1_{\wedge^{g-1} V^\vee \otimes \wedge^g V^{\vee\vee}} \otimes i_{\wedge^g V}\right) \circ \left(D^{1,g} \otimes \varphi_{i,g-i}\right) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \\
&= \rho \cdot \left(1_V \otimes ev^{g,\tau}_{V^\vee,a} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(D_{g-1,g} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}\right) \circ \left(1_{\wedge^{g-1} V^\vee \otimes \wedge^g V^{\vee\vee}} \otimes i_{\wedge^g V}\right) \\
&\quad \circ \left(D^{1,g} \otimes \varphi_{i,g-i}\right) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \\
&= \rho \cdot \left(1_V \otimes ev^{g,\tau}_{V^\vee,a} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(D_{g-1,g} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}\right) \circ \left(1_{\wedge^{g-1} V^\vee \otimes \wedge^g V^{\vee\vee}} \otimes i_{\wedge^g V}\right) \\
&\quad \circ \left(D^{1,g} \otimes 1_{\wedge^g V}\right) \circ \left(1_V \otimes \varphi_{i,g-i}\right) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \\
&= \rho \cdot \left(1_V \otimes ev^{g,\tau}_{V^\vee,a} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(D_{g-1,g} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}\right) \circ \left(D^{1,g} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \\
&\quad \circ \left(1_V \otimes i_{\wedge^g V}\right) \circ \left(1_V \otimes \varphi_{i,g-i}\right) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \ \text{(by subsequent (35))} \\
&= (-1)^{(g-1)} \binom{g}{g-1}^{-1} \binom{r-1}{g-1} \rho \cdot \left(1_V \otimes i_{\wedge^g V}\right) \circ \left(1_V \otimes \varphi_{i,g-i}\right) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \\
&= (-1)^{(g-1)} \binom{g}{g-1}^{-1} \binom{r-1}{g-1} \rho \cdot \left(1_V \otimes i_{\wedge^g V}\right) \circ \tau_{\wedge^g V, V} \circ \left(\varphi_{i,g-i} \otimes 1_V\right) \\
&= (-1)^{(g-1)} \binom{g}{g-1}^{-1} \binom{r-1}{g-1} \rho \cdot \tau_{\wedge^g V^{\vee\vee}, V} \circ \left(i_{\wedge^g V} \otimes 1_V\right) \circ \left(\varphi_{i,g-i} \otimes 1_V\right),
\end{aligned}$$

[3] The morphism denoted by $\varphi^{13}_{g-i,i-1}$ (resp. $\varphi^{13}_{i,g-i-1}$) in Proposition 2.3 is the one here denoted by $\varphi^{13}_{g-i,i-1} \otimes 1_{\wedge^g V^{\vee\vee}}$ (resp. $\varphi^{13}_{i,g-i-1} \otimes 1_{\wedge^g V^{\vee\vee}}$).

where we have employed the equality

$$\left(1_V \otimes ev^{g,\tau}_{V^\vee,a}\right) \circ \left(D_{g-1,g} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ D^{1,g} = (-1)^{(g-1)} \binom{g}{g-1}^{-1} \binom{r-1}{g-1} \tag{35}$$

from Theorem 2.1 (1). $\square$

We now consider the following morphisms. We have

$$\psi^V_{i,g-i-1} : \wedge^i V \otimes \wedge^{g-i-1} V \overset{\varphi_{i,g-i-1}}{\to} \wedge^{g-1} V \overset{D^{g-1,g}}{\to} V^\vee \otimes \wedge^g V^{\vee\vee}$$

and

$$\begin{aligned} \overline{\psi}^V_{g-i,1} \quad : \quad & \wedge^{g-i} V \otimes V^\vee \overset{D^{g-i,g} \otimes 1_{V^\vee}}{\to} \wedge^i V^\vee \otimes \wedge^g V^{\vee\vee} \otimes V^\vee \overset{\varphi^{13}_{i,1}}{\to} \wedge^{i+1} V^\vee \otimes \wedge^g V^{\vee\vee} \\ & \overset{D_{i+1,g} \otimes 1_{\wedge^g V^{\vee\vee}}}{\to} \wedge^{g-i-1} V \otimes \wedge^g V^\vee \otimes \wedge^g V^{\vee\vee} \overset{1_{\wedge^{g-i-1}V} \otimes ev^{g,\tau}_{V^\vee,a}}{\to} \wedge^{g-i-1} V. \end{aligned}$$

On the other hand we have

$$\psi^V_{g-i,i-1} : \wedge^{g-i} V \otimes \wedge^{i-1} V \overset{\varphi_{g-i,i-1}}{\to} \wedge^{g-1} V \overset{D^{g-1,g}}{\to} V^\vee \otimes \wedge^g V^{\vee\vee}$$

and

$$\begin{aligned} \overline{\psi}^V_{i,1} \quad : \quad & \wedge^{i} V \otimes V^\vee \overset{D^{i,g} \otimes 1_{V^\vee}}{\to} \wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee} \otimes V^\vee \overset{\varphi^{13}_{g-i,1}}{\to} \wedge^{g-i+1} V^\vee \otimes \wedge^g V^{\vee\vee} \\ & \overset{D_{g-i+1,g} \otimes 1_{\wedge^g V^{\vee\vee}}}{\to} \wedge^{i-1} V \otimes \wedge^g V^\vee \otimes \wedge^g V^{\vee\vee} \overset{1_{\wedge^{i-1}V} \otimes ev^{g,\tau}_{V^\vee,a}}{\to} \wedge^{i-1} V. \end{aligned}$$

The proof of the following result is a bit more involved and we will leave some of the details to the reader.

Lemma 4.2. *Setting*

$$\begin{aligned} \rho^{g-i,i}_{V^\vee} &:= (-1)^{(g-1)} \binom{g}{g-1}^{-1} \binom{g}{g-i}^{-1} \binom{g}{i}^{-1} \binom{r-1}{g-1} \binom{r-i}{g-i} \binom{r+i-g}{i} g, \\ \nu^{i,1}_{V^\vee} &:= (-1)^{(i+1)(g-i)}\, r_{\wedge^g V} \binom{g}{i}^{-1} \binom{r+i-g}{i} i \text{ and} \\ \nu^{g-i,1}_{V^\vee} &:= (-1)^i \binom{g}{g-i}^{-1} \binom{r-i}{g-i} (g-i)\,, \end{aligned}$$

the following diagram is commutative:

$$\begin{tikzcd} \wedge^{g-i} V \otimes \wedge^i V \otimes V^\vee \arrow[rr, "\left(1_{\wedge^{g-i}V} \otimes \overline{\psi}_{i,1}, \left(1_{\wedge^i V} \otimes \overline{\psi}_{g-i,1}\right) \circ \left(\tau_{\wedge^{g-i}V, \wedge^i V} \otimes 1_{V^\vee}\right)\right)"] \arrow[d, "\varphi_{g-i,i} \otimes 1_{V^\vee}"'] & & \wedge^{g-i} V \otimes \wedge^{i-1} V \oplus \wedge^i V \otimes \wedge^{g-i-1} V \arrow[d, "\nu^{i,1}_{V^\vee} \cdot \psi_{g-i,i-1} \oplus \nu^{g-i,1}_{V^\vee} \cdot \psi_{i,g-i-1}"] \\ \wedge^g V \otimes V^\vee \arrow[rr, "\rho^{g-i,i}_{V^\vee} \cdot (1_{V^\vee} \otimes i_{\wedge^g V}) \circ \tau_{\wedge^g V, V^\vee}"] & & V^\vee \otimes \wedge^g V^{\vee\vee}. \end{tikzcd}$$

Proof. By definition

$$\begin{aligned} \psi^V_{g-i,i-1} \circ \left(1_{\wedge^i V} \otimes \overline{\psi}^V_{i,1}\right) &= D^{g-1,g} \circ \varphi_{g-i,i-1} \circ \left(1_{\wedge^{g-i}V \otimes \wedge^{i-1}V} \otimes ev^{g,\tau}_{V^\vee,a}\right) \\ &\circ \left(1_{\wedge^{g-i}V} \otimes D_{g-i+1,g} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(1_{\wedge^{g-i}V} \otimes \varphi^{13}_{g-i,1}\right) \circ \left(1_{\wedge^{g-i}V} \otimes D^{i,g} \otimes 1_{V^\vee}\right), & (36) \\ \psi^V_{i,g-i-1} \circ \left(1_{\wedge^i V} \otimes \overline{\psi}^V_{g-i,1}\right) \circ \left(\tau_{\wedge^{g-i}V,\wedge^i V} \otimes 1_{V^\vee}\right) &= D^{g-1,g} \circ \varphi_{i,g-i-1} \circ \left(1_{\wedge^i V \otimes \wedge^{g-i-1}V} \otimes ev^{g,\tau}_{V^\vee,a}\right) \\ &\circ \left(1_{\wedge^i V} \otimes D_{i+1,g} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(1_{\wedge^i V} \otimes \varphi^{13}_{i,1}\right) \circ \left(1_{\wedge^i V} \otimes D^{g-i,g} \otimes 1_{V^\vee}\right) \circ \left(\tau_{\wedge^{g-i}V,\wedge^i V} \otimes 1_{V^\vee}\right). & (37) \end{aligned}$$

It follows from Theorem 2.1 (1) that we have, setting $\mu_{g-i,g} := \binom{g}{g-i}^{-1}\binom{r-i}{g-i}$ and $\mu_{i,g} := \binom{g}{i}^{-1}\binom{r+i-g}{i}$:

$$(-1)^{i(g-i)} \mu_{i,g} \cdot 1_{\wedge^{g-i}V} = \left(1_{\wedge^{g-i}V} \otimes ev^{g,\tau}_{V^\vee,a}\right) \circ \left(D_{i,g} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ D^{g-i,g}, \tag{38}$$

$$(-1)^{i(g-i)} \mu_{g-i,g} \cdot 1_{\wedge^i V} = \left(1_{\wedge^i V} \otimes ev^{g,\tau}_{V^\vee,a}\right) \circ \left(D_{g-i,g} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ D^{i,g}. \tag{39}$$

Inserting (38) in the definition (36), one checks that:

$$\begin{aligned}&(-1)^{i(g-i)}\,(-1)^{g-i}\,\mu_{i,g} i\cdot\psi^V_{g-i,i-1}\circ\left(1_{\wedge^i V}\otimes\overline{\psi}^V_{i,1}\right)\\&=D^{g-1,g}\circ\left(1_{\wedge^{g-1}V}\otimes ev^{g,\tau}_{V^\vee,a}\otimes ev^{g,\tau}_{V^\vee,a}\right)\circ a\circ\left(D^{g-i,g}\otimes D^{i,g}\otimes 1_{V^\vee}\right),\end{aligned}\tag{40}$$

where

$$\begin{aligned}a:=&(-1)^{g-i}\,i\cdot\left(\varphi_{g-i,i-1}\otimes 1_{\wedge^g V^\vee\otimes\wedge^g V^{\vee\vee}\otimes\wedge^g V^\vee\otimes\wedge^g V^{\vee\vee}}\right)\\&\circ\left(1_{\wedge^{g-i}V}\otimes\tau_{\wedge^g V^\vee\otimes\wedge^g V^{\vee\vee},\wedge^{i-1}V}\otimes 1_{\wedge^g V^\vee\otimes\wedge^g V^{\vee\vee}}\right)\\&\circ\left(D_{i,g}\otimes 1_{\wedge^g V^{\vee\vee}}\otimes D_{g-i+1,g}\otimes 1_{\wedge^g V^{\vee\vee}}\right)\circ\left(1_{\wedge^i V^\vee\otimes\wedge^g V^{\vee\vee}}\otimes\varphi^{13}_{g-i,1}\right)\end{aligned}$$

Similarly, inserting (39) in the definition (37), one finds:

$$\begin{aligned}&(-1)^{i(g-i)}\,(-1)^{i(g-i-1)}\,\mu_{g-i,g}\,(g-i)\cdot\psi^V_{i,g-i-1}\circ\left(1_{\wedge^i V}\otimes\overline{\psi}^V_{g-i,1}\right)\circ\left(\tau_{\wedge^{g-i}V,\wedge^i V}\otimes 1_{V^\vee}\right)\\&=D^{g-1,g}\circ\left(1_{\wedge^{g-1}V}\otimes ev^{g,\tau}_{V^\vee,a}\otimes ev^{g,\tau}_{V^\vee,a}\right)\circ b\circ\left(D^{g-i,g}\otimes D^{i,g}\otimes 1_{V^\vee}\right)\end{aligned}\tag{41}$$

where

$$\begin{aligned}b:=&(-1)^{i(g-i-1)}\,(g-i)\cdot\left(\varphi_{i,g-i-1}\otimes 1_{\wedge^g V^\vee\otimes\wedge^g V^{\vee\vee}\otimes\wedge^g V^\vee\otimes\wedge^g V^{\vee\vee}}\right)\\&\circ\left(1_{\wedge^i V}\otimes\tau_{\wedge^g V^\vee\otimes\wedge^g V^{\vee\vee},\wedge^{g-i-1}V}\otimes 1_{\wedge^g V^\vee\otimes\wedge^g V^{\vee\vee}}\right)\circ\left(D_{g-i,g}\otimes 1_{\wedge^g V^{\vee\vee}}\otimes D_{i+1,g}\otimes 1_{\wedge^g V^{\vee\vee}}\right)\\&\circ\left(1_{\wedge^{g-i}V^\vee\otimes\wedge^g V^{\vee\vee}}\otimes\varphi^{13}_{i,1}\right)\circ\left(\tau_{\wedge^i V^\vee\otimes\wedge^g V^{\vee\vee},\wedge^{g-i}V^\vee\otimes\wedge^g V^{\vee\vee}}\otimes 1_{V^\vee}\right)\end{aligned}$$

and we have used a similar commutative diagram in the last equality.

By Proposition 2.3 (second diagram) we have, setting $\rho:=r_{\wedge^g V}\, g\binom{g}{g-i}^{-1}\binom{r-i}{g-i}$,

$$\rho\cdot\left(D_{1,g}\otimes\varphi_{i,g-i}\right)\circ\tau_{\wedge^i V^\vee\otimes\wedge^{g-i}V^\vee,V^\vee}=a_0+b_0,\tag{42}$$

where

$$\begin{aligned}a_0:=&(-1)^{g-i}\,i\cdot\left(\varphi_{g-i,i-1}\otimes 1_{\wedge^g V^\vee\otimes\wedge^g V^\vee}\right)\circ\left(1_{\wedge^{g-i}V}\otimes\tau_{\wedge^g V^\vee,\wedge^{i-1}V}\otimes 1_{\wedge^g V^\vee}\right)\\&\circ\left(D_{i,g}\otimes D_{g-i+1,g}\right)\circ\left(1_{\wedge^i V^\vee}\otimes\varphi_{g-i,1}\right),\\b_0:=&(-1)^{i(g-i-1)}\,(g-i)\cdot\left(\varphi_{i,g-i-1}\otimes 1_{\wedge^g V^\vee\otimes\wedge^g V^\vee}\right)\circ\left(1_{\wedge^i V}\otimes\tau_{\wedge^g V^\vee,\wedge^{g-i-1}V}\otimes 1_{\wedge^g V^\vee}\right)\\&\circ\left(D_{g-i,g}\otimes D_{i+1,g}\right)\circ\left(1_{\wedge^{g-i}V^\vee}\otimes\varphi_{i,1}\right)\circ\left(\tau_{\wedge^i V^\vee,\wedge^{g-i}V^\vee}\otimes 1_{V^\vee}\right).\end{aligned}$$

Consider the morphism

$$\tau_{(235)}:\wedge^i V^\vee\otimes\wedge^{g-i}V^\vee\otimes V^\vee\otimes\wedge^g V^{\vee\vee}\otimes\wedge^g V^{\vee\vee}\to\wedge^i V^\vee\otimes\wedge^g V^{\vee\vee}\otimes\wedge^{g-i}V^\vee\otimes\wedge^g V^{\vee\vee}\otimes V^\vee$$

attached to the permutation (235). After a tedious computation one can verify the following relations:

$$\tau_{(35)}\circ a\circ\tau_{(235)}=\tau_{(34)}\circ\left(a_0\otimes 1_{\wedge^g V^{\vee\vee}\otimes\wedge^g V^{\vee\vee}}\right),\tag{43}$$

$$b\circ\tau_{(235)}=\tau_{(34)}\circ\left(b_0\otimes 1_{\wedge^g V^{\vee\vee}\otimes\wedge^g V^{\vee\vee}}\right),\tag{44}$$

$$\tau_{(345)}\circ\left(D_{1,g}\otimes\varphi^{13}_{i,g-i}\right)\circ\tau_{\wedge^i V^\vee\otimes\wedge^g V^{\vee\vee}\otimes\wedge^{g-i}V^\vee\otimes\wedge^g V^{\vee\vee},V^\vee}\circ\tau_{(235)}=\tau_{(34)}\circ\left(c_0\otimes 1_{\wedge^g V^{\vee\vee}\otimes\wedge^g V^{\vee\vee}}\right)\tag{45}$$

Thanks to (43), (44) and (45), the equality (42) gives

$$\rho\cdot\tau_{(345)}\circ\left(D_{1,g}\otimes\varphi^{13}_{i,g-i}\right)\circ\tau_{\wedge^i V^\vee\otimes\wedge^g V^{\vee\vee}\otimes\wedge^{g-i}V^\vee\otimes\wedge^g V^{\vee\vee},V^\vee}=\tau_{(35)}\circ a+b.\tag{46}$$

Finally, we need to remark that we have the following commutative diagram (by a computation of the involved permutations):

$$\begin{tikzcd}\wedge^{g-1}V\otimes\wedge^g V^\vee\otimes\wedge^g V^{\vee\vee}\otimes\wedge^g V^\vee\otimes\wedge^g V^{\vee\vee} \arrow[r, "\tau_{(35)}"] \arrow[d, "1_{\wedge^{g-1}V\otimes\wedge^g V^\vee}\otimes\tau_{\wedge^g V^{\vee\vee},\wedge^g V^\vee}\otimes 1_{\wedge^g V^{\vee\vee}}"'] & \wedge^{g-1}V\otimes\wedge^g V^\vee\otimes\wedge^g V^{\vee\vee}\otimes\wedge^g V^\vee\otimes\wedge^g V^{\vee\vee} \arrow[d, "1_{\wedge^{g-1}V\otimes\wedge^g V^\vee}\otimes\tau_{\wedge^g V^{\vee\vee},\wedge^g V^\vee}\otimes 1_{\wedge^g V^{\vee\vee}}"] \\ \wedge^{g-1}V\otimes\wedge^g V^\vee\otimes\wedge^g V^\vee\otimes\wedge^g V^{\vee\vee}\otimes\wedge^g V^{\vee\vee} \arrow[r, "\tau_{\wedge^g V^{\vee\vee},\wedge^g V^{\vee\vee}}"] & \wedge^{g-1}V\otimes\wedge^g V^\vee\otimes\wedge^g V^\vee\otimes\wedge^g V^{\vee\vee}\otimes\wedge^g V^{\vee\vee}.\end{tikzcd}$$

Since $\wedge^g V^{\vee\vee}$ is invertible, it follows from [De3, 7.2 Lemme] that we have $\tau_{\wedge^g V^{\vee\vee},\wedge^g V^{\vee\vee}} = r_{\wedge^g V^{\vee\vee}} = r_{\wedge^g V}$ in the above diagram, implying that $\tau_{(35)} = r_{\wedge^g V}$ as well. Hence (46) becomes

$$\rho \cdot \tau_{(345)} \circ \left(D_{1,g} \otimes \varphi^{13}_{i,g-i}\right) \circ \tau_{\wedge^i V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee}, V^\vee} = r_{\wedge^g V} \cdot a + b. \tag{47}$$

We can now compute:

$$\begin{aligned}
&(-1)^{i(g-i)} (-1)^{g-i} r_{\wedge^g V} \mu_{i,g} i \cdot \psi^V_{g-i,i-1} \circ \left(1_{\wedge^i V} \otimes \overline{\psi}^V_{i,1}\right) \\
&+ (-1)^{i(g-i)} (-1)^{i(g-i-1)} \mu_{g-i,g} (g-i) \cdot \psi^V_{i,g-i-1} \circ \left(1_{\wedge^i V} \otimes \overline{\psi}^V_{g-i,1}\right) \\
&\quad \circ \left(\tau_{\wedge^{g-i} V, \wedge^i V} \otimes 1_{V^\vee}\right) \text{ (by (40) and (41))} \\
&= D^{g-1,g} \circ \left(1_{\wedge^{g-1} V} \otimes ev^{g,\tau}_{V^\vee,a} \otimes ev^{g,\tau}_{V^\vee,a}\right) \circ r_{\wedge^g V} \cdot a \circ \left(D^{g-i,g} \otimes D^{i,g} \otimes 1_{V^\vee}\right) \\
&\quad + D^{g-1,g} \circ \left(1_{\wedge^{g-1} V} \otimes ev^{g,\tau}_{V^\vee,a} \otimes ev^{g,\tau}_{V^\vee,a}\right) \circ b \circ \left(D^{g-i,g} \otimes D^{i,g} \otimes 1_{V^\vee}\right) \\
&= D^{g-1,g} \circ \left(1_{\wedge^{g-1} V} \otimes ev^{g,\tau}_{V^\vee,a} \otimes ev^{g,\tau}_{V^\vee,a}\right) \circ \left(r_{\wedge^g V} \cdot a + b\right) \circ \left(D^{g-i,g} \otimes D^{i,g} \otimes 1_{V^\vee}\right) \text{ (by (47))} \\
&= \rho \cdot D^{g-1,g} \circ \left(1_{\wedge^{g-1} V} \otimes ev^{g,\tau}_{V^\vee,a} \otimes ev^{g,\tau}_{V^\vee,a}\right) \circ \tau_{(345)} \circ \left(D_{1,g} \otimes \varphi^{13}_{i,g-i}\right) \\
&\quad \circ \tau_{\wedge^i V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee}, V^\vee} \circ \left(D^{g-i,g} \otimes D^{i,g} \otimes 1_{V^\vee}\right) \text{ (by a formal computation)} \\
&= \rho \cdot D^{g-1,g} \circ \left(1_{\wedge^{g-1} V} \otimes ev^{g,\tau}_{V^\vee,a} \otimes ev^{g,\tau}_{V^\vee,a}\right) \circ \left(D_{1,g} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^\vee \otimes \wedge^g V^{\vee\vee}}\right) \circ \tau_{(234)} \\
&\quad \circ \left(1_{V^\vee} \otimes \varphi^{13}_{i,g-i}\right) \circ \left(1_{V^\vee} \otimes D^{g-i,g} \otimes D^{i,g}\right) \circ \tau_{\wedge^{g-i} V \otimes \wedge^i V, V^\vee}.
\end{aligned} \tag{48}$$

We remark that we have, by definition, $r_{\wedge^g V^\vee} = ev^g_{V^\vee,a} \circ \tau_{\wedge^g V^\vee, \wedge^g V^{\vee\vee}} \circ C_{\wedge^g V^\vee}$ and, since $\wedge^g V^\vee$ is invertible, $r_{\wedge^g V} = r_{\wedge^g V^\vee} = r^{-1}_{\wedge^g V^\vee}$ and we deduce $\left(ev^g_{V^\vee,a}\right)^{-1} = r_{\wedge^g V} \cdot \tau_{\wedge^g V^\vee, \wedge^g V^{\vee\vee}} \circ C_{\wedge^g V^\vee}$. This gives the first of the subsequent equalities, while the second follows from a standard property of the Casimir element:

$$\begin{aligned}
&\left(1_{\wedge^g V^{\vee\vee}} \otimes ev^{g,\tau}_{V^\vee,a}\right) \circ \left(\left(ev^g_{V^\vee,a}\right)^{-1} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \\
&\quad = r_{\wedge^g V} \cdot \left(1_{\wedge^g V^{\vee\vee}} \otimes ev^{g,\tau}_{V^\vee,a}\right) \circ \left(\tau_{\wedge^g V^\vee, \wedge^g V^{\vee\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(C_{\wedge^g V^\vee} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \\
&\quad = r_{\wedge^g V} \cdot 1_{\wedge^g V^{\vee\vee}}.
\end{aligned} \tag{49}$$

Thanks to Theorem 2.1 (1), we know that $\left(1_{V^\vee} \otimes ev^g_{V^\vee}\right) \circ \left(D^{g-1,g} \otimes 1_{\wedge^g V^\vee}\right) \circ D_{1,g} = (-1)^{(g-1)} \mu_{g-1,g}$ with $\mu_{g-1,g} := \binom{g}{g-1}^{-1} \binom{r-1}{g-1}$. Employing this relation in the second of the subsequent equalities, we find

$$\begin{aligned}
&D^{g-1,g} \circ \left(1_{\wedge^{g-1} V} \otimes ev^{g,\tau}_{V^\vee,a} \otimes ev^{g,\tau}_{V^\vee,a}\right) \circ \left(D_{1,g} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^\vee \otimes \wedge^g V^{\vee\vee}}\right) \\
&= \left(1_{V^\vee \otimes \wedge^g V^{\vee\vee}} \otimes ev^{g,\tau}_{V^\vee,a} \otimes ev^{g,\tau}_{V^\vee,a}\right) \circ \left(D^{g-1,g} \otimes 1_{\wedge^g V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^g V^\vee \otimes \wedge^g V^{\vee\vee}}\right) \\
&\quad \circ \left(D_{1,g} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^\vee \otimes \wedge^g V^{\vee\vee}}\right) \\
&= (-1)^{(g-1)} \mu_{g-1,g} \cdot \left(1_{V^\vee \otimes \wedge^g V^{\vee\vee}} \otimes ev^{g,\tau}_{V^\vee,a} \otimes ev^{g,\tau}_{V^\vee,a}\right) \circ \left(1_{V^\vee} \otimes \left(ev^g_{V^\vee}\right)^{-1} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^\vee \otimes \wedge^g V^{\vee\vee}}\right) \\
&= (-1)^{(g-1)} \mu_{g-1,g} \cdot \left(1_{V^\vee \otimes \wedge^g V^{\vee\vee}} \otimes ev^{g,\tau}_{V^\vee,a}\right) \circ \left(1_{V^\vee} \otimes \left(ev^g_{V^\vee,a}\right)^{-1} \otimes 1_{\wedge^g V^{\vee\vee}} \otimes ev^{g,\tau}_{V^\vee,a}\right) \\
&= (-1)^{(g-1)} \mu_{g-1,g} \cdot \left(1_{V^\vee \otimes \wedge^g V^{\vee\vee}} \otimes ev^{g,\tau}_{V^\vee,a}\right) \circ \left(1_{V^\vee} \otimes \left(ev^g_{V^\vee,a}\right)^{-1} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \\
&\quad \circ \left(1_{V^\vee \otimes \wedge^g V^{\vee\vee}} \otimes ev^{g,\tau}_{V^\vee,a}\right) \text{ (by (49))} \\
&= (-1)^{(g-1)} \mu_{g-1,g} r_{\wedge^g V} \cdot 1_{V^\vee \otimes \wedge^g V^{\vee\vee}} \otimes ev^{g,\tau}_{V^\vee,a}.
\end{aligned} \tag{50}$$

We also have, thanks to the relation $\varphi_{i,g-i}^{13\to\wedge^g V^{\vee\vee}} \circ \left(D^{g-i,g} \otimes D^{i,g}\right) = \mu_{i,g} \cdot i_{\wedge^g V} \circ \varphi_{g-i,i}$ with $\mu_{i,g} := \binom{g}{i}^{-1}\binom{r+i-g}{i}$ arising from Theorem 2.1 (2):

$$\begin{aligned}
&\left(1_{V^\vee \otimes \wedge^g V^{\vee\vee}} \otimes ev_{V^\vee,a}^{g,\tau}\right) \circ \tau_{(234)} \circ \left(1_{V^\vee} \otimes \varphi_{i,g-i}^{13}\right) \circ \left(1_{V^\vee} \otimes D^{g-i,g} \otimes D^{i,g}\right) \\
&= \left(1_{V^\vee} \otimes ev_{V^\vee,a}^{g,\tau} \otimes 1_{\wedge^g V^{\vee\vee}}\right) \circ \left(1_{V^\vee} \otimes \varphi_{i,g-i}^{13}\right) \circ \left(1_{V^\vee} \otimes D^{g-i,g} \otimes D^{i,g}\right) \\
&= \left(1_{V^\vee} \otimes \varphi_{i,g-i}^{13\to\wedge^g V^{\vee\vee}}\right) \circ \left(1_{V^\vee} \otimes D^{g-i,g} \otimes D^{i,g}\right) = \mu_{i,g} \cdot \left(1_{V^\vee} \otimes i_{\wedge^g V}\right) \circ \left(1_{V^\vee} \otimes \varphi_{g-i,i}\right). \qquad (51)
\end{aligned}$$

Inserting (50) and (51) in (48) gives the claim after a small computation. □

4.2. **Laplace and Dirac operators.** We now specialize the above discussion to the case $g = 2i$, i.e. $i = g-i$, and we simply write L for the invertible object $\wedge^g V^{\vee\vee}$ and set $L^{-1} := \wedge^g V^\vee$. We write $\mathrm{Alt}^n(M) := \wedge^n M$ and $\mathrm{Sym}^n(M) := \vee^n M$ when M is an alternating power of V. Attached to the multiplication map $\wedge^i V \otimes \wedge^i V \overset{\varphi_{i,i}}{\to} \wedge^g V \overset{i_{\wedge^g V}}{\to} L$ there are the Laplace operators

$$\Delta^n_{i_{\wedge^g V}\circ\varphi_{i,i},a} : \mathrm{Alt}^n\left(\wedge^i V\right) \to \mathrm{Alt}^{n-2}\left(\wedge^i V\right) \otimes L, \qquad \Delta^n_{i_{\wedge^g V}\circ\varphi_{i,i},s} : \mathrm{Sym}^n\left(\wedge^i V\right) \to \mathrm{Sym}^{n-2}\left(\wedge^i V\right) \otimes L$$

and, since $\varphi_{i,i} \circ \tau_{\wedge^i V,\wedge^i V} = (-1)^i \varphi_{i,i}$, by Lemma 3.1 we have $\Delta^n_{i_{\wedge^g V}\circ\varphi_{i,i},s} = 0$ (resp. $\Delta^n_{i_{\wedge^g V}\circ\varphi_{i,i},a} = 0$) when i is odd (resp. i is even). Hence, we set $\Delta^n := \Delta^n_{i_{\wedge^g V}\circ\varphi_{i,i},a}$ (resp. $\Delta^n := \Delta^n_{i_{\wedge^g V}\circ\varphi_{i,i},s}$) when i is odd (resp. even).

Looking at the pairings defined before Lemma 4.1, we note that we have $\psi^V_{i,1} = \psi^V_{g-i,1}$ and $\overline{\psi}^V_{g-i,g-i-1} = \overline{\psi}^V_{i,i-1}$, while looking at the pairings defined before 4.2, we remark the equalities $\psi^V_{i,g-i-1} = \psi^V_{g-i,i-1}$ and $\overline{\psi}^V_{g-i,1} = \overline{\psi}^V_{i,1}$.

Suppose first that i is odd. Then we define the following Dirac operators, for every integer $n \geq 1$:

$$\begin{array}{llll}
\partial_1^n := \partial^n_{\psi^V_{g-i,1},a} : & \mathrm{Alt}^n\left(\wedge^i V\right) \otimes V & \to & \mathrm{Alt}^{n-1}\left(\wedge^i V\right) \otimes \wedge^{i-1} V^\vee \otimes L, \\
\overline{\partial}_{i-1}^n := \partial^n_{\overline{\psi}^V_{i,i-1},a} : & \mathrm{Alt}^n\left(\wedge^i V\right) \otimes \wedge^{i-1} V^\vee & \to & \mathrm{Alt}^{n-1}\left(\wedge^i V\right) \otimes V, \\
\partial_{i-1}^n := \partial^n_{\psi^V_{g-i,i-1},a} : & \mathrm{Alt}^n\left(\wedge^i V\right) \otimes \wedge^{i-1} V & \to & \mathrm{Alt}^{n-1}\left(\wedge^i V\right) \otimes V^\vee \otimes L, \\
\overline{\partial}_1^n := \partial^n_{\overline{\psi}^V_{i,1},a} : & \mathrm{Alt}^n\left(\wedge^i V\right) \otimes V^\vee & \to & \mathrm{Alt}^{n-1}\left(\wedge^i V\right) \otimes \wedge^{i-1} V.
\end{array}$$

Theorem 4.3. *Suppose that i is odd, so that $\Delta^n : \mathrm{Alt}^n\left(\wedge^i V\right) \to \mathrm{Alt}^{n-2}\left(\wedge^i V\right) \otimes L$ and set*

$$\rho^i := (-1)^{i+1} r_L \binom{g}{g-1}^{-1} \binom{g}{i}^{-1} \binom{r-1}{g-1} \binom{r-i}{i} \frac{g}{i} = r_L \binom{g}{g-1}^{-1} \binom{g}{i}^{-1} \binom{r-1}{g-1} \binom{r-i}{i} \frac{g}{i}.$$

(1) *The following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{Sym}^n\left(\wedge^i V\right) \otimes V \arrow[r, "\partial_1^n"] \arrow[d, "\Delta^n \otimes 1_V"'] & \mathrm{Sym}^{n-1}\left(\wedge^i V\right) \otimes \wedge^{i-1} V^\vee \otimes L \arrow[d, "\overline{\partial}_{i-1}^{n-1} \otimes 1_L"] \\
\mathrm{Sym}^{n-2}\left(\wedge^i V\right) \otimes L \otimes V \arrow[r, "\frac{\rho^i}{2} \cdot 1_{\mathrm{Sym}^{n-2}(\wedge^i V)} \otimes \tau_{L,V}"'] & \mathrm{Sym}^{n-2}\left(\wedge^i V\right) \otimes V \otimes L.
\end{tikzcd}$$

(2) *When $r_L = 1$, the first of the following diagrams is commutative and it becomes equivalent to the second diagram when we further assume that $\binom{r-i}{i} \in End(\mathbb{I})$ is a non-zero divisor:*

$$\begin{tikzcd}
\mathrm{Sym}^n\left(\wedge^i V\right) \otimes V^\vee \arrow[r, "\overline{\partial}_1^n"] \arrow[d, "\Delta^n \otimes 1_{V^\vee}"'] & \mathrm{Sym}^{n-1}\left(\wedge^i V\right) \otimes \wedge^{i-1} V \arrow[d, "\binom{r-i}{i} \cdot \partial_{i-1}^{n-1}"] \\
\mathrm{Sym}^{n-2}\left(\wedge^i V\right) \otimes L \otimes V^\vee \arrow[r, "\binom{r-i}{i} \frac{\rho^i}{2} \cdot 1_{\mathrm{Sym}^{n-2}(\wedge^i V)} \otimes \tau_{L,V^\vee}"'] & \mathrm{Sym}^{n-2}\left(\wedge^i V\right) \otimes V^\vee \otimes L,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Sym}^n\left(\wedge^i V\right) \otimes V^\vee \arrow[r, "\overline{\partial}_1^n"] \arrow[d, "\Delta^n \otimes 1_{V^\vee}"'] & \mathrm{Sym}^{n-1}\left(\wedge^i V\right) \otimes \wedge^{i-1} V \arrow[d, "\partial_{i-1}^{n-1}"] \\
\mathrm{Sym}^{n-2}\left(\wedge^i V\right) \otimes L \otimes V^\vee \arrow[r, "\frac{\rho^i}{2} \cdot 1_{\mathrm{Sym}^{n-2}(\wedge^i V)} \otimes \tau_{L,V^\vee}"'] & \mathrm{Sym}^{n-2}\left(\wedge^i V\right) \otimes V^\vee \otimes L.
\end{tikzcd}$$

(3) *Suppose that $L \simeq \mathbb{L}^{\otimes 2}$ for some invertible object $\mathbb{L}$, that $r_{\wedge^i V} < 0$ (see definition 3.6) and that V has alternating rank g. Then there are morphisms*

$$\begin{aligned} s_{\Delta}^{n-2} &: \mathrm{Alt}^{n-2}\left(\wedge^i V\right) \otimes L \to \mathrm{Alt}^n\left(\wedge^i V\right) \text{ for } n \geq 2, \\ s_{\overline{\partial}_{i-1}}^{n-1} &: \mathrm{Alt}^{n-1}\left(\wedge^i V\right) \otimes V \to \mathrm{Alt}^n\left(\wedge^i V\right) \otimes \wedge^{i-1} V^{\vee} \text{ for } n \geq 1, \\ s_{\partial_{i-1}}^{n-1} &: \mathrm{Alt}^{n-1}\left(\wedge^i V\right) \otimes V^{\vee} \otimes L \to \mathrm{Alt}^n\left(\wedge^i V\right) \otimes \wedge^{i-1} V \text{ for } n \geq 1 \end{aligned}$$

such that

$$\Delta^n \circ s_{\Delta}^{n-2} = 1_{\mathrm{Alt}^{n-2}(\wedge^i V)\otimes L}, \quad \overline{\partial}_{i-1}^n \circ s_{\overline{\partial}_{i-1}}^{n-1} = 1_{\mathrm{Alt}^{n-1}(\wedge^i V)\otimes V}$$
$$\text{and} \quad \partial_{i-1}^n \circ s_{\partial_{i-1}}^{n-1} = 1_{\mathrm{Alt}^{n-1}(\wedge^i V)\otimes V^{\vee}\otimes L}.$$

In particular, the following objects exist:

$$\ker\left(\Delta^n\right) \subset \mathrm{Alt}^n\left(\wedge^i V\right), \quad \ker\left(\overline{\partial}_{i-1}^n\right) \subset \mathrm{Alt}^n\left(\wedge^i V\right) \otimes \wedge^{i-1} V^{\vee}, \quad \ker\left(\partial_{i-1}^n\right) \subset \mathrm{Alt}^n\left(\wedge^i V\right) \otimes \wedge^{i-1} V.$$

Proof. (1-2) Looking at the quantities $\nu_V^{g-i,1}$ and $\nu_V^{i,1}$ (resp. $\nu_{V^\vee}^{i,1}$ and $\nu_{V^\vee}^{g-i,1}$) from Lemma 4.1 (resp. Lemma 4.2) when $i = g - i$, we see that $\nu_V^{g-i,1} = (-1)^i \cdot \nu_V^{i,1}$ (resp. $\nu_{V^\vee}^{i,1} = (-1)^i r_{\wedge^g V} \cdot \nu_{V^\vee}^{g-i,1}$). Since i is odd, it follows that $\nu_V^{g-i,1} = -\nu_V^{i,1}$ (resp. $\nu_{V^\vee}^{i,1} = -r_{\wedge^g V} \cdot \nu_{V^\vee}^{g-i,1}$). We have that $i_{\wedge^g V} \circ \varphi_{i,i}$ is alternating, so that we may apply Lemma 3.3 to deduce the claimed commutativity in (1) (resp. the first commutative diagram in (2) when $r_L = r_{\wedge^g V} = 1$): we have indeed $\nu_V^{g-i,1} \in \mathbb{Q}^\times$ and

$$\rho_V^{i,g-i}/\nu_V^{g-i,1} = \rho^i \text{ (resp. } \rho_{V^\vee}^{g-i,i} = -r_{\wedge^g V}\binom{g}{i}^{-1}\binom{r-i}{i} i\rho^i \text{ and } \nu_{V^\vee}^{i,1} = r_{\wedge^g V}\binom{g}{i}^{-1}\binom{r-i}{i} i)$$

and the commutativity of (2) is deduced simplifying by $r_{\wedge^g V}\binom{g}{i}^{-1} i$. If $\binom{r-i}{i} \in End\left(\mathbb{I}\right)$ is a non-zero divisor we may further simplify to get the second commutative diagram in (2).

(3) Indeed $L \simeq \mathbb{L}^{\otimes 2}$ implies $r_L = 1$ and, since V has alternating rank g, by Corollary 2.2 $i_{\wedge^g V} \circ \varphi_{i,i}$ is a perfect alternating pairing. Since $r_{\wedge^i V} < 0$, Lemma 3.12 gives the existence of s_Δ^{n-2} and $\ker\left(\Delta^n\right)$. We also remark that, since $\binom{r-i}{g-i} = \binom{r-i}{i} \in End\left(\mathbb{I}\right)$ and $\binom{r-1}{g-1} \in End\left(\mathbb{I}\right)$ are invertible (once again because V has alternating rank g), it follows that $\pm\frac{\rho^i}{2}$ is invertible, that $\alpha := \frac{\rho^i}{2} \cdot \left(1_{\mathrm{Alt}^{n-2}(\wedge^i V)} \otimes \tau_{L,V}\right)$ is an isomorphism and, hence, that $f := \alpha \circ \left(\Delta^n \otimes 1_V\right)$ has a section $s := \left(s_\Delta^{n-2} \otimes 1_V\right) \circ \alpha^{-1}$ such that

$$f \circ s = \alpha \circ \left(\Delta^n \otimes 1_V\right) \circ \left(s_\Delta^{n-2} \otimes 1_V\right) \circ \alpha^{-1} = \alpha \circ \alpha^{-1} = 1_{\mathrm{Alt}^{n-2}(\wedge^i V)\otimes V^\vee \otimes L}.$$

Similarly $f := \left(-\frac{\rho^i}{2} \cdot 1_{\mathrm{Alt}^{n-2}(\wedge^i V)} \otimes \tau_{L,V^\vee}\right) \circ \left(\Delta^n \otimes 1_{V^\vee}\right)$ has a section. We can now apply the following simple remark to the commutative diagram in (1) (resp. the second commutative diagram in (2)). Suppose that we are given

$$f : X \xrightarrow{f_1} Y \xrightarrow{f_2} Z$$

and that $s : Z \to X$ is a morphism such that $f \circ s = 1_Z$. Then, setting $s_2 := f_1 \circ s$, we see that

$$f_2 \circ s_2 = f_2 \circ f_1 \circ s = f \circ s = 1_Z,$$

implying that f_2 has a section. But then there is an associated idempotent $e_2 := s_2 \circ f_2$ and $\ker\left(f_2\right) = \ker\left(e_2\right)$ exists because V is pseudo-abelian. This gives the existence of a section of $\overline{\partial}_{i-1}^n \otimes 1_L$, hence of $\overline{\partial}_{i-1}^n$ and $\ker\left(\overline{\partial}_{i-1}^n\right)$ because L is invertible, and of $s_{\partial_{i-1}}^{n-1}$ and $\ker\left(\partial_{i-1}\right)$. $\square$

Suppose now that i is even. Then we define the following Dirac operators, for every integer $n \geq 1$:

$$\begin{aligned} \partial_1^n := \partial^n_{\psi^V_{g-i,1},s} &: \mathrm{Sym}^n\left(\wedge^i V\right) \otimes V &\to \mathrm{Sym}^{n-1}\left(\wedge^i V\right) \otimes \wedge^{i-1} V^\vee \otimes L, \\ \overline{\partial}_{i-1}^n := \partial^n_{\overline{\psi}^V_{i,i-1},s} &: \mathrm{Sym}^n\left(\wedge^i V\right) \otimes \wedge^{i-1} V^\vee &\to \mathrm{Sym}^{n-1}\left(\wedge^i V\right) \otimes V, \\ \partial_{i-1}^n := \partial^n_{\psi^V_{g-i,i-1},s} &: \mathrm{Sym}^n\left(\wedge^i V\right) \otimes \wedge^{i-1} V &\to \mathrm{Sym}^{n-1}\left(\wedge^i V\right) \otimes V^\vee \otimes L, \\ \overline{\partial}_1^n := \partial^n_{\overline{\psi}^V_{i,1},s} &: \mathrm{Sym}^n\left(\wedge^i V\right) \otimes V^\vee &\to \mathrm{Sym}^{n-1}\left(\wedge^i V\right) \otimes \wedge^{i-1} V. \end{aligned}$$

Theorem 4.4. *Suppose that i is even, so that $\Delta^n : \mathrm{Sym}^n\left(\wedge^i V\right) \to \mathrm{Sym}^{n-2}\left(\wedge^i V\right) \otimes L$ and set*

$$\begin{aligned}\rho^i &:= (-1)^{i+1} r_L \binom{g}{g-1}^{-1} \binom{g}{i}^{-1} \binom{r-1}{g-1} \binom{r-i}{i} \frac{g}{i} \\ &= -r_L \binom{g}{g-1}^{-1} \binom{g}{i}^{-1} \binom{r-1}{g-1} \binom{r-i}{i} \frac{g}{i}.\end{aligned}$$

(1) *The following diagram is commutative:*

$$\begin{tikzcd} \mathrm{Sym}^n\left(\wedge^i V\right) \otimes V \arrow[r, "\partial_1^n"] \arrow[d, "\Delta^n \otimes 1_V"'] & \mathrm{Sym}^{n-1}\left(\wedge^i V\right) \otimes \wedge^{i-1} V^\vee \otimes L \arrow[d, "\overline{\partial}_{i-1}^n \otimes 1_L"] \\ \mathrm{Sym}^{n-2}\left(\wedge^i V\right) \otimes L \otimes V \arrow[r, "\frac{\rho^i}{2} \cdot 1_{\mathrm{Sym}^{n-2}(\wedge^i V)} \otimes \tau_{L,V}"'] & \mathrm{Sym}^{n-2}\left(\wedge^i V\right) \otimes V \otimes L. \end{tikzcd}$$

(2) *When $r_L = 1$, the first of the following diagrams is commutative and it becomes equivalent to the second diagram when we further assume that $\binom{r-i}{i} \in End(\mathbb{I})$ is a non-zero divisor:*

$$\begin{tikzcd} \mathrm{Sym}^n\left(\wedge^i V\right) \otimes V^\vee \arrow[r, "\overline{\partial}_1^n"] \arrow[d, "\Delta^n \otimes 1_{V^\vee}"'] & \mathrm{Sym}^{n-1}\left(\wedge^i V\right) \otimes \wedge^{i-1} V \arrow[d, "\binom{r-i}{i} \cdot \partial_{i-1}^n"] \\ \mathrm{Sym}^{n-2}\left(\wedge^i V\right) \otimes L \otimes V^\vee \arrow[r, "\binom{r-i}{i} \frac{\rho^i}{2} \cdot 1_{\mathrm{Sym}^{n-2}(\wedge^i V)} \otimes \tau_{L,V^\vee}"'] & \mathrm{Sym}^{n-2}\left(\wedge^i V\right) \otimes V^\vee \otimes L, \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Sym}^n\left(\wedge^i V\right) \otimes V^\vee \arrow[r, "\overline{\partial}_1^n"] \arrow[d, "\Delta^n \otimes 1_{V^\vee}"'] & \mathrm{Sym}^{n-1}\left(\wedge^i V\right) \otimes \wedge^{i-1} V \arrow[d, "\partial_{i-1}^n"] \\ \mathrm{Sym}^{n-2}\left(\wedge^i V\right) \otimes L \otimes V^\vee \arrow[r, "\frac{\rho^i}{2} \cdot 1_{\mathrm{Sym}^{n-2}(\wedge^i V)} \otimes \tau_{L,V^\vee}"'] & \mathrm{Sym}^{n-2}\left(\wedge^i V\right) \otimes V^\vee \otimes L. \end{tikzcd}$$

(3) *Suppose that $L \simeq \mathbb{L}^{\otimes 2}$ for some invertible object $\mathbb{L}$, that $r_{\wedge^i V} > 0$ (see definition 3.6) and that V has alternating rank g. Then there are morphisms*

$$\begin{aligned} s_\Delta^{n-2} &: \mathrm{Sym}^{n-2}\left(\wedge^i V\right) \otimes L \to \mathrm{Sym}^n\left(\wedge^i V\right) \text{ for } n \geq 2, \\ s_{\overline{\partial}_{i-1}}^{n-1} &: \mathrm{Sym}^{n-1}\left(\wedge^i V\right) \otimes V \to \mathrm{Sym}^n\left(\wedge^i V\right) \otimes \wedge^{i-1} V^\vee \text{ for } n \geq 1, \\ s_{\partial_{i-1}}^{n-1} &: \mathrm{Sym}^{n-1}\left(\wedge^i V\right) \otimes V^\vee \otimes L \to \mathrm{Sym}^n\left(\wedge^i V\right) \otimes \wedge^{i-1} V \text{ for } n \geq 1 \end{aligned}$$

such that

$$\begin{aligned} &\Delta^n \circ s_\Delta^{n-2} = 1_{\mathrm{Sym}^{n-2}(\wedge^i V) \otimes L}, \overline{\partial}_{i-1}^n \circ s_{\overline{\partial}_{i-1}}^{n-1} = 1_{\mathrm{Sym}^{n-1}(\wedge^i V) \otimes V} \\ &\text{and } \partial_{i-1}^n \circ s_{\partial_{i-1}}^{n-1} = 1_{\mathrm{Sym}^{n-1}(\wedge^i V) \otimes V^\vee \otimes L}. \end{aligned}$$

In particular, the following objects exist:

$$\begin{aligned} &\ker\left(\Delta^n\right) \subset \mathrm{Sym}^n\left(\wedge^i V\right), \\ &\ker\left(\overline{\partial}_{i-1}^n\right) \subset \mathrm{Sym}^n\left(\wedge^i V\right) \otimes \wedge^{i-1} V^\vee, \\ &\ker\left(\partial_{i-1}^n\right) \subset \mathrm{Sym}^n\left(\wedge^i V\right) \otimes \wedge^{i-1} V. \end{aligned}$$

Proof. (1-2) As in the proof of Theorem 4.3 we have $\nu_V^{g-i,1} = (-1)^i \cdot \nu_V^{i,1}$ (resp. $\nu_{V^\vee}^{i,1} = (-1)^i r_{\wedge^g V} \cdot \nu_{V^\vee}^{g-i,1}$). Since i is even, it follows that $\nu_V^{g-i,1} = \nu_V^{i,1}$ (resp. $\nu_{V^\vee}^{i,1} = r_{\wedge^g V} \cdot \nu_{V^\vee}^{g-i,1}$). Then the proof is identical to the proof of Theorem 4.3, noticing that we have once again $\rho_V^{i,g-i}/\nu_V^{g-i,1} = \rho^i$ and $\nu_{V^\vee}^{i,1} = r_{\wedge^g V} \binom{g}{i}^{-1} \binom{r-i}{i} i$, but now $\rho_{V^\vee}^{g-i,i} = r_{\wedge^g V} \binom{g}{i}^{-1} \binom{r-i}{i} i \rho^i$, justifying the change of sign in the second commutative diagram of (2) with respect to that of Theorem 4.3.

(3) The proof is identical to the proof of Theorem 4.3, noticing that here we need to assume $r_{\wedge^i V} > 0$ in order to apply Lemma 3.12 because now $i_{\wedge^g V} \circ \varphi_{i,i}$ is a perfect symmetric pairing. ☐

5. Laplace and Dirac operators for the symmetric algebras

In this section we assume that we are given an object $V \in \mathcal{C}$ such that $\vee^g V$ is invertible. If X is an object we set $r_X := \mathrm{rank}(X)$, so that $r_{\vee^g V} \in \{\pm 1\}$, and we use the shorthand $r := r_V$.

5.1. **Preliminary lemmas.** We define

$$\psi^V_{i,1} : \vee^i V \otimes V \overset{\varphi_{i,1}}{\to} \vee^{i+1} V \overset{D^{i+1,g}}{\to} \vee^{g-i-1} V^\vee \otimes \vee^g V^{\vee\vee},$$

and

$$\begin{aligned}\overline{\psi}^V_{g-i,g-i-1} \quad : \quad & \vee^{g-i} V \otimes \vee^{g-i-1} V^\vee \overset{D^{g-i,g} \otimes 1_{\vee^{g-i-1} V^\vee}}{\to} \vee^i V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^{g-i-1} V^\vee \overset{\varphi^{13}_{i,g-i-1}}{\to} \vee^{g-1} V^\vee \otimes \vee^g V^{\vee\vee} \\ & \overset{D_{g-1,g} \otimes 1_{\vee^g V^{\vee\vee}}}{\to} V \otimes \vee^g V^\vee \otimes \vee^g V^{\vee\vee} \overset{1_V \otimes ev^{g,\tau}_{V^\vee,a}}{\to} V.\end{aligned}$$

We may also consider

$$\psi^V_{g-i,1} : \vee^{g-i} V \otimes V \overset{\varphi_{g-i,1}}{\to} \vee^{g-i+1} V \overset{D^{g-i+1,g}}{\to} \vee^{i-1} V^\vee \otimes \vee^g V^{\vee\vee},$$

and

$$\begin{aligned}\overline{\psi}^V_{i,i-1} \quad : \quad & \vee^i V \otimes \vee^{i-1} V^\vee \overset{D^{i,g} \otimes 1_{\vee^{i-1} V^\vee}}{\to} \vee^{g-i} V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^{i-1} V^\vee \overset{\varphi^{13}_{g-i,i-1}}{\to} \vee^{g-1} V^\vee \otimes \vee^g V^{\vee\vee} \\ & \overset{D_{g-1,g} \otimes 1_{\vee^g V^{\vee\vee}}}{\to} V \otimes \vee^g V^\vee \otimes \vee^g V^{\vee\vee} \overset{1_V \otimes ev^{g,\tau}_{V^\vee,a}}{\to} V.\end{aligned}$$

Lemma 5.1. *Setting*

$$\begin{aligned}&\rho^{i,g-i}_V := r_{\vee^g V} \binom{g}{g-1}^{-1} \binom{g}{g-i}^{-1} \binom{r+g-1}{g-1} \binom{r+g-1}{g-i} g, \\ &\nu^{g-i,1}_V := i \text{ and } \nu^{i,1}_V := g-i\end{aligned}$$

the following diagram is commutative:

$$\begin{array}{ccc}\vee^i V \otimes \vee^{g-i} V \otimes V & \xrightarrow{\left(1_{\vee^i V} \otimes \psi_{g-i,1}, \left(1_{\vee^{g-i} V} \otimes \psi_{i,1}\right) \circ \left(\tau_{\vee^i V, \vee^{g-i} V} \otimes 1_V\right)\right)} & \vee^i V \otimes \vee^{i-1} V^\vee \otimes \vee^g V^{\vee\vee} \oplus \vee^{g-i} V \otimes \vee^{g-i-1} V^\vee \otimes \vee^g V^{\vee\vee} \\ \Big\downarrow {\scriptstyle \varphi_{i,g-i} \otimes 1_V} & & \Big\downarrow {\scriptstyle \nu^{g-i,1}_V \cdot \overline{\psi}_{i,i-1} \otimes 1_{\vee^g V^{\vee\vee}} \oplus \nu^{i,1}_V \cdot \overline{\psi}_{g-i,g-i-1} \otimes 1_{\vee^g V^{\vee\vee}}} \\ \vee^g V \otimes V & \xrightarrow{\rho^{i,g-i}_V \cdot \tau_{\vee^g V^{\vee\vee}, V} \circ (i_{\vee^g V} \otimes 1_V)} & V \otimes \vee^g V^{\vee\vee}.\end{array}$$

Proof. The proof is just a copy of that of Lemma 4.1, replacing the use of Theorem 2.1 (resp. Proposition 2.3) with Theorem 2.4 (resp. Proposition 2.6). ☐

We now consider the following morphisms. We have

$$\psi^V_{i,g-i-1} : \vee^i V \otimes \vee^{g-i-1} V \overset{\varphi_{i,g-i-1}}{\to} \vee^{g-1} V \overset{D^{g-1,g}}{\to} V^\vee \otimes \vee^g V^{\vee\vee}$$

and

$$\begin{aligned}\overline{\psi}^V_{g-i,1} \quad : \quad & \vee^{g-i} V \otimes V^\vee \overset{D^{g-i,g} \otimes 1_{V^\vee}}{\to} \vee^i V^\vee \otimes \vee^g V^{\vee\vee} \otimes V^\vee \overset{\varphi^{13}_{i,1}}{\to} \vee^{i+1} V^\vee \otimes \vee^g V^{\vee\vee} \\ & \overset{D_{i+1,g} \otimes 1_{\vee^g V^{\vee\vee}}}{\to} \vee^{g-i-1} V \otimes \vee^g V^\vee \otimes \vee^g V^{\vee\vee} \overset{1_{\vee^{g-i-1} V} \otimes ev^{g,\tau}_{V^\vee,a}}{\to} \vee^{g-i-1} V.\end{aligned}$$

On the other hand we have

$$\psi^V_{g-i,i-1} : \vee^{g-i} V \otimes \vee^{i-1} V \overset{\varphi_{g-i,i-1}}{\to} \vee^{g-1} V \overset{D^{g-1,g}}{\to} V^\vee \otimes \vee^g V^{\vee\vee}$$

and

$$\begin{aligned}\overline{\psi}^V_{i,1} \quad : \quad & \vee^i V \otimes V^\vee \overset{D^{i,g} \otimes 1_{V^\vee}}{\to} \vee^{g-i} V^\vee \otimes \vee^g V^{\vee\vee} \otimes V^\vee \overset{\varphi^{13}_{g-i,1}}{\to} \vee^{g-i+1} V^\vee \otimes \vee^g V^{\vee\vee} \\ & \overset{D_{g-i+1,g} \otimes 1_{\vee^g V^{\vee\vee}}}{\to} \vee^{i-1} V \otimes \vee^g V^\vee \otimes \vee^g V^{\vee\vee} \overset{1_{\vee^{i-1} V} \otimes ev^{g,\tau}_{V^\vee,a}}{\to} \vee^{i-1} V.\end{aligned}$$

Lemma 5.2. *Setting*

$$\begin{aligned}
\rho^{g-i,i}_{V^\vee} &:= \binom{g}{g-1}^{-1}\binom{g}{g-i}^{-1}\binom{g}{i}^{-1}\binom{r+g-1}{g-1}\binom{r+g-1}{g-i}\binom{r+g-1}{i}g,\\
\nu^{i,1}_{V^\vee} &:= r_{\vee^g V}\binom{g}{i}^{-1}\binom{r+g-1}{i}i \text{ and}\\
\nu^{g-i,1}_{V^\vee} &:= \binom{g}{g-i}^{-1}\binom{r+g-1}{g-i}(g-i),
\end{aligned}$$

the following diagram is commutative:

$$\begin{tikzcd}
\vee^{g-i}V\otimes\vee^i V\otimes V^\vee \arrow[r, "\left(1_{\vee^{g-i}V}\otimes\overline{\psi}_{i,1},\left(1_{\vee^i V}\otimes\overline{\psi}_{g-i,1}\right)\circ\left(\tau_{\vee^{g-i}V,\vee^i V}\otimes 1_{V^\vee}\right)\right)"] \arrow[d, "\varphi_{g-i,i}\otimes 1_{V^\vee}"'] & \vee^{g-i}V\otimes\vee^{i-1}V\oplus\vee^i V\otimes\vee^{g-i-1}V \arrow[d, "\nu^{i,1}_{V^\vee}\cdot\psi_{g-i,i-1}\oplus\nu^{g-i,1}_{V^\vee}\cdot\psi_{i,g-i-1}"] \\
\vee^g V\otimes V^\vee \arrow[r, "\rho^{g-i,i}_{V^\vee}\cdot\left(1_{V^\vee}\otimes i_{\vee^g V}\right)\circ\tau_{\vee^g V,V^\vee}"] & V^\vee\otimes\vee^g V^{\vee\vee}.
\end{tikzcd}$$

Proof. Again the proof is a copy of that of Lemma 4.2. ∎

5.2. **Laplace and Dirac operators.** We now specialize the above discussion to the case $g=2i$, i.e. $i=g-i$, and we simply write L for the invertible object $\vee^g V^{\vee\vee}$ and set $L^{-1}:=\vee^g V^\vee$. We write $\mathrm{Alt}^n(M):=\wedge^n M$ and $\mathrm{Sym}^n(M):=\vee^n M$ when M is a symmetric power of V. Attached to the multiplication map $\vee^i V\otimes\vee^i V\overset{\varphi_{i,i}}{\to}\vee^g V\overset{i_{\vee^g V}}{\to}L$ there are the Laplace operators

$$\begin{aligned}
\Delta^n_{i_{\vee^g V}\circ\varphi_{i,i},a} &: \mathrm{Alt}^n\left(\vee^i V\right)\to\mathrm{Alt}^{n-2}\left(\vee^i V\right)\otimes L,\\
\Delta^n_{i_{\vee^g V}\circ\varphi_{i,i},s} &: \mathrm{Sym}^n\left(\vee^i V\right)\to\mathrm{Sym}^{n-2}\left(\vee^i V\right)\otimes L
\end{aligned}$$

and, since $\varphi_{i,i}\circ\tau_{\vee^i V,\vee^i V}=\varphi_{i,i}$, by Lemma 3.1 we have $\Delta^n_{i_{\vee^g V}\circ\varphi_{i,i},a}=0$. Hence we will only consider $\Delta^n:=\Delta^n_{i_{\vee^g V}\circ\varphi_{i,i},s}$.

Looking at the pairings defined before Lemma 5.1, we note that we have $\psi^V_{i,1}=\psi^V_{g-i,1}$ and $\overline{\psi}^V_{g-i,g-i-1}=\overline{\psi}^V_{i,i-1}$, while looking at the pairings defined before 5.2, we remark the equalities $\psi^V_{i,g-i-1}=\psi^V_{g-i,i-1}$ and $\overline{\psi}^V_{g-i,1}=\overline{\psi}^V_{i,1}$.

Then we define the following Dirac operators, for every integer $n\geq 1$:

$$\begin{array}{llll}
\partial^n_1:=\partial^n_{\psi^V_{g-i,1},s}: & \mathrm{Sym}^n\left(\vee^i V\right)\otimes V & \to & \mathrm{Sym}^{n-1}\left(\vee^i V\right)\otimes\vee^{i-1}V^\vee\otimes L,\\
\overline{\partial}^n_{i-1}:=\partial^n_{\overline{\psi}^V_{i,i-1},s}: & \mathrm{Sym}^n\left(\vee^i V\right)\otimes\vee^{i-1}V^\vee & \to & \mathrm{Sym}^{n-1}\left(\vee^i V\right)\otimes V,\\
\partial^n_{i-1}:=\partial^n_{\psi^V_{g-i,i-1},s}: & \mathrm{Sym}^n\left(\vee^i V\right)\otimes\vee^{i-1}V & \to & \mathrm{Sym}^{n-1}\left(\vee^i V\right)\otimes V^\vee\otimes L,\\
\overline{\partial}^n_1:=\partial^n_{\overline{\psi}^V_{i,1},s}: & \mathrm{Sym}^n\left(\vee^i V\right)\otimes V^\vee & \to & \mathrm{Sym}^{n-1}\left(\vee^i V\right)\otimes\vee^{i-1}V.
\end{array}$$

In the same way as we have deduced Theorem 4.3 from Lemmas 4.1 and 4.2, the following result can be deduced from Lemmas 5.1 and 5.2.

Theorem 5.3. *Set*

$$\rho^i:=r_{\vee^g V}\binom{g}{g-1}^{-1}\binom{g}{i}^{-1}\binom{r+g-1}{g-1}\binom{r+g-1}{i}\frac{g}{i}.$$

(1) *The following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{Sym}^n\left(\vee^i V\right)\otimes V \arrow[r, "\partial^n_1"] \arrow[d, "\Delta^n\otimes 1_V"'] & \mathrm{Sym}^{n-1}\left(\vee^i V\right)\otimes\vee^{i-1}V^\vee\otimes L \arrow[d, "\overline{\partial}^{n-1}_{i-1}\otimes 1_L"] \\
\mathrm{Sym}^{n-2}\left(\vee^i V\right)\otimes L\otimes V \arrow[r, "\frac{\rho^i}{2}\cdot 1_{\mathrm{Sym}^{n-2}\left(\vee^i V\right)}\otimes\tau_{L,V}"'] & \mathrm{Sym}^{n-2}\left(\vee^i V\right)\otimes V\otimes L.
\end{tikzcd}$$

(2) *When $r_L = 1$, the first of the following diagrams is commutative and it becomes equivalent to the second diagram when we further assume that $\binom{r+g-1}{i} \in End(\mathbb{I})$ is a non-zero divisor:*

$$\begin{tikzcd}
\mathrm{Sym}^n\left(\vee^i V\right) \otimes V^\vee \arrow[r, "\overline{\partial}_1^n"] \arrow[d, "\Delta^n \otimes 1_{V^\vee}"'] & \mathrm{Sym}^{n-1}\left(\vee^i V\right) \otimes \vee^{i-1} V \arrow[d, "\binom{r+g-1}{i}\cdot\partial_{i-1}^{n-1}"] \\
\mathrm{Sym}^{n-2}\left(\vee^i V\right) \otimes L \otimes V^\vee \arrow[r, "\binom{r+g-1}{i}\frac{\rho^i}{2}\cdot 1_{\mathrm{Sym}^{n-2}(\vee^i V)} \otimes \tau_{L,V^\vee}"'] & \mathrm{Sym}^{n-2}\left(\vee^i V\right) \otimes V^\vee \otimes L,
\end{tikzcd}
\quad
\begin{tikzcd}
\mathrm{Sym}^n\left(\vee^i V\right) \otimes V^\vee \arrow[r, "\overline{\partial}_1^n"] \arrow[d, "\Delta^n \otimes 1_{V^\vee}"'] & \mathrm{Sym}^{n-1}\left(\vee^i V\right) \otimes \vee^{i-1} V \arrow[d, "\partial_{i-1}^{n-1}"] \\
\mathrm{Sym}^{n-2}\left(\vee^i V\right) \otimes L \otimes V^\vee \arrow[r, "\frac{\rho^i}{2}\cdot 1_{\mathrm{Sym}^{n-2}(\vee^i V)} \otimes \tau_{L,V^\vee}"'] & \mathrm{Sym}^{n-2}\left(\vee^i V\right) \otimes V^\vee \otimes L.
\end{tikzcd}$$

(3) *Suppose that $L \simeq \mathbb{L}^{\otimes 2}$ for some invertible object $\mathbb{L}$, that $r_{\vee^i V} > 0$ (see definition 3.6) and that V has symmetric rank g. Then there are morphisms*

$$\begin{aligned}
s_\Delta^{n-2} &: \mathrm{Sym}^{n-2}\left(\vee^i V\right) \otimes L \to \mathrm{Sym}^n\left(\vee^i V\right) \text{ for } n \geq 2, \\
s_{\overline{\partial}_{i-1}}^{n-1} &: \mathrm{Sym}^{n-1}\left(\vee^i V\right) \otimes V \to \mathrm{Sym}^n\left(\vee^i V\right) \otimes \vee^{i-1} V^\vee \text{ for } n \geq 1, \\
s_{\partial_{i-1}}^{n-1} &: \mathrm{Sym}^{n-1}\left(\vee^i V\right) \otimes V^\vee \otimes L \to \mathrm{Sym}^n\left(\vee^i V\right) \otimes \vee^{i-1} V \text{ for } n \geq 1
\end{aligned}$$

such that

$$\Delta^n \circ s_\Delta^{n-2} = 1_{\mathrm{Sym}^{n-2}(\vee^i V)\otimes L},\ \overline{\partial}_{i-1}^n \circ s_{\overline{\partial}_{i-1}}^{n-1} = 1_{\mathrm{Sym}^{n-1}(\vee^i V)\otimes V}$$
$$\text{and } \partial_{i-1}^n \circ s_{\partial_{i-1}}^{n-1} = 1_{\mathrm{Sym}^{n-1}(\vee^i V)\otimes V^\vee \otimes L}.$$

In particular, the following objects exist:

$$\begin{aligned}
&\ker\left(\Delta^n\right) \subset \mathrm{Sym}^n\left(\vee^i V\right), \\
&\ker\left(\overline{\partial}_{i-1}^n\right) \subset \mathrm{Sym}^n\left(\vee^i V\right) \otimes \vee^{i-1} V^\vee, \\
&\ker\left(\partial_{i-1}^n\right) \subset \mathrm{Sym}^n\left(\vee^i V\right) \otimes \vee^{i-1} V.
\end{aligned}$$

6. Some remarks about the functoriality of the Dirac operators

We will assume, from now on, that we are given an object $V \in \mathcal{C}$ and that $\mathcal{C}$ and $\mathcal{D}$ are $\mathbb{Q}$-linear rigid and pseudo-abelian *ACU* tensor categories. Once again, if X is an object, we set $r_X := \mathrm{rank}(X)$ and we use the shorthand $r := r_V$. As usual, we write $e_{X,?}^n$, $i_{X,?}^n$ and $p_{X,?}^n$ for the idempotent $e_{X,?}^n$ in $End(\otimes^n X)$ giving rise to $\wedge^n X$ when $? = a$ and $\vee^n X$ when $? = s$ and the associated canonical injective and surjective morphisms. We denote by $D_{V,?}^{i,j}$ and $D_{i,j}^{V,?}$ the Poincare duality morphisms in the algebra $\otimes^\cdot V$ when $? = t$, $\wedge^\cdot V$ when $? = a$ and $\vee^\cdot V$ when $? = s$. Then it easily follows from [MS, Lemma 2.3, §5 and §6] that, for every $g \geq i$ and $? = a$ or s,

$$D_{V,?}^{i,g} : A_i \overset{i_{V,?}^i}{\to} \otimes^i V \overset{D_t^{i,g}}{\to} \left(\otimes^{g-i} V^\vee\right) \otimes \left(\otimes^g V^{\vee\vee}\right) \overset{p_{V^\vee,?}^{g-i} \otimes p_{V^{\vee\vee},?}^g}{\to} A_{g-i}^\vee \otimes A_g^{\vee\vee} \tag{52}$$

and

$$D_{i,g}^{V,?} : A_i^\vee \overset{i_{V,?}^i}{\to} \otimes^i V^\vee \overset{D_{i,g}^t}{\to} \left(\otimes^{g-i} V\right) \otimes \left(\otimes^g V^\vee\right) \overset{p_{V,?}^{g-i} \otimes p_{V^\vee,?}^g}{\to} A_{g-i} \otimes A_g^\vee \tag{53}$$

Suppose that we are given a (covariant) additive *AU* tensor functor $F : \mathcal{C} \to \mathcal{D}$; it preserves internal homs and dualities. We suppose that F has the following further properties:

- $F(\tau_{V,V}) = \varepsilon \cdot \tau_{F(V),F(V)}$ and $F(\tau_{V^\vee,V^\vee}) = \varepsilon \cdot \tau_{F(V),F(V)}$, where $\varepsilon \in \{\pm 1\}$;
- $F(\tau_{V^\vee,V}) = \eta \cdot \tau_{F(V)^\vee,F(V)}$ (so that $F(\tau_{V,V^\vee}) = \eta \cdot \tau_{F(V),F(V)^\vee}$), where $\eta \in \{\pm 1\}$.

We remark that, if $\varepsilon = 1$ (resp. $\varepsilon = -1$) and $X \in \{V, V^\vee, V^{\vee\vee}\}$, we have $F\left(e_{X,a}^n\right) = e_{F(X),a}^n$ (resp. $F\left(e_{X,a}^n\right) = e_{F(X),s}^n$), $F\left(e_{X,s}^n\right) = e_{F(X),s}^n$ (resp. $F\left(e_{X,s}^n\right) = e_{F(X),a}^n$) and the same for the associated injective and surjective morphisms. The following result is now an easy consequence of this remark, (52), (53) and an explicit computation showing that $F\left(D_{V,t}^{i,g}\right) = \eta^{\frac{i^2+i}{2}} \cdot D_{F(V),t}^{i,g}$ and $F\left(D_{i,g}^{V,t}\right) = \eta^{\frac{i^2-i}{2}} \cdot D_{i,g}^{F(V),t}$ (see [MS, §5] for the explicit of $D_{V,t}^{i,j}$ and $D_{i,j}^{V,t}$).

Lemma 6.1. *Suppose that we are given a (covariant) additive AU tensor functor $F : \mathcal{C} \to \mathcal{D}$ as above.*

(1) *If $\varepsilon = 1$ then we have*

$$F\left(D^{i,g}_{V,a}\right) = \eta^{\frac{i^2+i}{2}} \cdot D^{i,g}_{F(V),a},\ F\left(D^{i,g}_{V,s}\right) = \eta^{\frac{i^2+i}{2}} \cdot D^{i,g}_{F(V),s},$$
$$F\left(D^{V,a}_{i,g}\right) = \eta^{\frac{i^2-i}{2}} \cdot D^{F(V),a}_{i,g},\ F\left(D^{V,s}_{i,g}\right) = \eta^{\frac{i^2-i}{2}} \cdot D^{F(V),s}_{i,g}.$$

(2) *If $\varepsilon = -1$ then the same formulas hold after swapping the symbols s and a in the right-hand side.*

Fix $g \geq i$ such that $g = 2i$ and set $L_a := \wedge^g V^{\vee\vee}$ and $L_s := \vee^g V^{\vee\vee}$. Write $\psi^{V,a}_{i,1} = \psi^{V,a}_{g-i,1}$ and $\overline{\psi}^{V,a}_{g-i,g-i-1} = \overline{\psi}^{V,a}_{i,i-1}$ for the pairings defined before Lemma 4.1, $\psi^{V,a}_{i,g-i-1} = \psi^{V,a}_{g-i,i-1}$ and $\overline{\psi}^{V,a}_{g-i,1} = \overline{\psi}^{V,a}_{i,1}$ for those defined before 4.2, $\psi^{V,s}_{i,1} = \psi^{V,s}_{g-i,1}$ and $\overline{\psi}^{V,s}_{g-i,g-i-1} = \overline{\psi}^{V,s}_{i,i-1}$ for the ones considered before Lemma 5.1 and $\psi^{V,s}_{i,g-i-1} = \psi^{V,s}_{g-i,i-1}$ and $\overline{\psi}^{V,s}_{g-i,1} = \overline{\psi}^{V,s}_{i,1}$ for those defined before 5.2.

Consider the following operators from §§4.2[4]:

$$\begin{array}{llll}
\Delta^{\mathrm{Alt}^n(\wedge^i V)} := \Delta^n_{i_{\wedge^g V}\circ\varphi_{i,i},a}: & \mathrm{Alt}^n\left(\wedge^i V\right) & \to & \mathrm{Alt}^{n-2}\left(\wedge^i V\right)\otimes L_a, \\
\overline{\partial}^{\mathrm{Alt}^n(\wedge^i V)}_{i-1} := \partial^n_{\overline{\psi}^{V,a}_{i,i-1},a}: & \mathrm{Alt}^n\left(\wedge^i V\right)\otimes\wedge^{i-1}V^\vee & \to & \mathrm{Alt}^{n-1}\left(\wedge^i V\right)\otimes V, \\
\partial^{\mathrm{Alt}^n(\wedge^i V)}_{i-1} := \partial^n_{\psi^{V,a}_{g-i,i-1},a}: & \mathrm{Alt}^n\left(\wedge^i V\right)\otimes\wedge^{i-1}V & \to & \mathrm{Alt}^{n-1}\left(\wedge^i V\right)\otimes V^\vee\otimes L_a,
\end{array}$$

and similar for $\Delta^{\mathrm{Sym}^n(\wedge^i V)} := \Delta^n_{i_{\wedge^g V}\circ\varphi_{i,i},a}$, $\overline{\partial}^{\mathrm{Sym}^n(\wedge^i V)}_{i-1} := \partial^n_{\overline{\psi}^{V,a}_{i,i-1},s}$ and $\partial^{\mathrm{Sym}^n(\wedge^i V)}_{i-1} := \partial^n_{\psi^{V,a}_{g-i,i-1},s}$ where one swaps the symbols Alt with Sym.

Similarly, in order to symmetrically state the results, we will need to consider the operators from §§5.2 together with the analogous operators induced on the alternating powers:

$$\begin{array}{llll}
\Delta^{\mathrm{Alt}^n(\vee^i V)} := \Delta^n_{i_{\vee^g V}\circ\varphi_{i,i},a}: & \mathrm{Alt}^n\left(\vee^i V\right) & \to & \mathrm{Alt}^{n-2}\left(\vee^i V\right)\otimes L_s, \\
\overline{\partial}^{\mathrm{Alt}^n(\vee^i V)}_{i-1} := \partial^n_{\overline{\psi}^{V,s}_{i,i-1},a}: & \mathrm{Alt}^n\left(\vee^i V\right)\otimes\vee^{i-1}V^\vee & \to & \mathrm{Alt}^{n-1}\left(\vee^i V\right)\otimes V, \\
\partial^{\mathrm{Alt}^n(\vee^i V)}_{i-1} := \partial^n_{\psi^{V,s}_{g-i,i-1},a}: & \mathrm{Alt}^n\left(\vee^i V\right)\otimes\vee^{i-1}V & \to & \mathrm{Alt}^{n-1}\left(\vee^i V\right)\otimes V^\vee\otimes L_s,
\end{array}$$

and similar for the remaining three operators with Alt and Sym swapped. The following result, whose proof is left to the reader, follows from Lemma 6.1 and a small computation.

Proposition 6.2. *Suppose that we are given a (covariant) additive AU tensor functor $F : \mathcal{C} \to \mathcal{D}$ as above.*

(1) *If $\varepsilon = 1$ then we have*

$$\begin{array}{ll}
F\left(\Delta^{\mathrm{Alt}^n(\wedge^i V)}\right) = \Delta^{\mathrm{Alt}^n(\wedge^i F(V))}, & F\left(\Delta^{\mathrm{Alt}^n(\vee^i V)}\right) = \Delta^{\mathrm{Alt}^n(\vee^i F(V))}, \\
F\left(\overline{\partial}^{\mathrm{Alt}^n(\wedge^i V)}_{i-1}\right) = \eta^{\frac{i(i+1)}{2}+1}\cdot\overline{\partial}^{\mathrm{Alt}^n(\wedge^i F(V))}_{i-1}, & F\left(\overline{\partial}^{\mathrm{Alt}^n(\vee^i V)}_{i-1}\right) = \eta^{\frac{i(i+1)}{2}+1}\cdot\overline{\partial}^{\mathrm{Alt}^n(\vee^i F(V))}_{i-1}, \\
F\left(\partial^{\mathrm{Alt}^n(\wedge^i V)}_{i-1}\right) = \eta^{\frac{g(g-1)}{2}}\cdot\partial^{\mathrm{Alt}^n(\wedge^i F(V))}_{i-1}, & F\left(\partial^{\mathrm{Alt}^n(\vee^i V)}_{i-1}\right) = \eta^{\frac{g(g-1)}{2}}\cdot\partial^{\mathrm{Alt}^n(\vee^i F(V))}_{i-1}.
\end{array}$$

Six more formulas hold where the symbols Alt *and* Sym *are swapped.*

(2) *If $\varepsilon = -1$ and i is even, then similar twelve formulas hold where in the right-hand side the symbols $\wedge$ and $\vee$ must be swapped. If $\varepsilon = -1$ and i is odd, in addition one must swap the symbols* Alt *and* Sym*.*

[4]Of course some of them will be zero, but it will be convenient to consider all of them, in order to state the result in a symmetric way.

6.1. **Application to quaternionic objects.** We will now focus on the case $i = 2$ and $g = 2i = 4$ and we let B be a quaternion $\mathbb{Q}$-algebra, whose main involution we denote by $b \mapsto b^{\iota}$. An alternating (resp. symmetric) quaternionic object in $\mathcal{C}$ is a couple (V, θ) where V has alternating (resp. symmetric) rank 4 and $\theta : B \to End(V)$ is a unitary ring homomorphism. We will assume that such a (V, θ) has been given in the following discussion.

We have $\vee^2 B \subset B \otimes B$, the $\mathbb{Q}$-vector space generated by the elements $b_1 \vee b_2 = \frac{1}{2}(b_1 \otimes b_2 + b_2 \otimes b_1)$. Noticing that

$$(b_1 + b_2) \otimes (b_1 + b_2) = b_1 \otimes b_1 + b_2 \otimes b_2 + b_1 \otimes b_2 + b_2 \otimes b_1$$

and that $b \vee b = b \otimes b$, we see that

$$b_1 \vee b_2 = \frac{(b_1 + b_2) \vee (b_1 + b_2)}{2} - \frac{b_1 \vee b_1}{2} - \frac{b_2 \vee b_2}{2},$$

so that $\vee^2 B$ is the $\mathbb{Q}$-vector space generated by the elements $b \vee b$. Considering $B \otimes B$ as a $\mathbb{Q}$-algebra in the natural way, it follows that $\vee^2 B$ is a subalgebra, because the product of elements of the form $b \vee b$ is again of this form. Let Tr: $B \to \mathbb{Q}$ and Nr: $B \to \mathbb{Q}$ be the reduced trace and norm and set $B_0 := \ker(\mathrm{Tr})$.

Write W for the $\mathbb{Q}$-vector space B, endowed with the action of $B \otimes B$ defined by the rule $b_1 \otimes b_2 \cdot x := b_1 x b_2^{\iota}$. It gives rise to a unitary ring homomorphism $f : B \otimes B \to End_{\mathbb{Q}}(W) \simeq \mathbf{M}_4(\mathbb{Q})$ which is injective because $B \otimes B$ is simple, hence an isomorphism by counting dimensions. As a $\vee^2 B$-module $W = B_0 \oplus \mathbb{Q}$ and it easily follows that the resulting homomorphism

$$\vee^2 B \to End_{\mathbb{Q}}(B_0) \oplus End_{\mathbb{Q}}(\mathbb{Q}) \simeq \mathbf{M}_3(\mathbb{Q}) \oplus \mathbb{Q}$$

is an isomorphism: it is injective because $End_{\mathbb{Q}}(B_0) \oplus End_{\mathbb{Q}}(\mathbb{Q}) \subset End_{\mathbb{Q}}(W)$ and f is injective, hence an isomorphism again by counting dimensions. Furthermore, the action of $\vee^2 B$ on $\mathbb{Q}$ is given by the $\mathbb{Q}$-algebra homomorphism

$$\chi : \vee^2 B \to \mathbb{Q},\ \chi(b_1 \vee b_2) = \frac{\mathrm{Tr}(b_1^{\iota} b_2)}{2}.$$

It follows that there is an idempotent $e_- \in \vee^2 B$ characterized by $s e_- = \chi(s) e_-$ for every $s \in \vee^2 B$.

We have a natural $B \otimes B$-action on $V \otimes V$ by $\theta^{\otimes 2} := \theta \otimes \theta$ and, since $b \otimes b \circ e_{V,?}^2 = e_{V,?}^2 \circ b \otimes b$ for $? \in \{a, s\}$, $B \subset B \otimes B$ (diagonally) operates on $\wedge^2 V$ and $\vee^2 V$. But $\vee^2 B$ is generated by the elements of the form $b \otimes b$ as a $\mathbb{Q}$-algebra (and indeed as a $\mathbb{Q}$-vector space, as already noticed): hence $\vee^2 B \subset B \otimes B$ operates on $\wedge^2 V$ and $\vee^2 V$. The above discussion shows that we may write

$$\wedge^2 V = \left(\wedge^2 V\right)_+ \oplus \left(\wedge^2 V\right)_- \text{ and } \vee^2 V = \left(\vee^2 V\right)_+ \oplus \left(\vee^2 V\right)_- \tag{54}$$

where $\left(\wedge^2 V\right)_- := \mathrm{Im}\left(\theta^{\otimes 2}(e_-)\right)$ and $\left(\vee^2 V\right)_- := \mathrm{Im}\left(\theta^{\otimes 2}(e_-)\right)$ are characterized by the property that $\vee^2 B$ acts on them via χ. Indeed we remark that, since $\vee^2 B$ is generated by the diagonal image of $B \subset B \otimes B$ and $\chi(b \otimes b) = \mathrm{Nr}(b)$, $\left(\wedge^2 V\right)_-$ (resp. $\left(\vee^2 V\right)_-$) is the unique maximal subobject of $\wedge^2 V$ (resp. $\vee^2 V$) on which B acts via the reduced norm.

Associated with (V, θ) is the dual quaternionic object $\left(V^\vee, \theta^\vee\right)$ where $\theta^\vee(b) := \theta(b^\iota)^\vee$. We will simply write $\left(\wedge^2 V^\vee\right)_\pm$ (resp. $\left(\vee^2 V^\vee\right)_\pm$) for the $\pm$ components attached to $\left(V^\vee, \theta^\vee\right)$ obtained in this way. Since by definition $\theta^{\vee \otimes 2}(e_-) := \left(\theta(e_-)^\vee\right)^{\otimes 2} = \left(\theta(e_-)^{\otimes 2}\right)^\vee$, we have $\left(\wedge^2 V^\vee\right)_\pm = \left(\wedge^2 V_\pm\right)^\vee$ (resp. $\left(\vee^2 V^\vee\right)_\pm = \left(\vee^2 V_\pm\right)^\vee$)

We summarize the above discussion in the first part of following lemma, while the second follows from the remark before Lemma 6.1.

Lemma 6.3. *If (V, θ) is an alternating (resp. symmetric) quaternionic object in $\mathcal{C}$, there is a canonical decomposition (54) (in the category of quaternionic objects), where $\left(\wedge^2 V\right)_-$ (resp. $\left(\vee^2 V\right)_-$) is characterized by the fact that it is the unique maximal subobject X of $\wedge^2 V$ (resp. $\vee^2 V$) such that the action of B acting diagonally on $\wedge^2 V$ (resp. $\vee^2 V$) is given by the reduced norm on X. We have $\left(\wedge^2 V^\vee\right)_\pm = \left(\wedge^2 V_\pm\right)^\vee$ (resp. $\left(\vee^2 V^\vee\right)_\pm = \left(\vee^2 V_\pm\right)^\vee$).*

Suppose that we are given a (covariant) additive AU tensor functor $F : \mathcal{C} \to \mathcal{D}$ as above and define $F(\theta)(b) := F(\theta(b))$. Then $(F(V), F(\theta))$ is an alternating (resp. symmetric) quaternionic object in $\mathcal{D}$

when $\varepsilon = 1$, $(F(V), F(\theta))$ is a symmetric (resp. alternating) quaternionic object in $\mathcal{D}$ when $\varepsilon = -1$ and we have

$$F\left(\left(\wedge^2 V\right)_{\pm}\right) = \left(\wedge^2 F(V)\right)_{\pm} \text{ (resp. } F\left(\left(\vee^2 V\right)_{\pm}\right) = \left(\vee^2 F(V)\right)_{\pm}\text{) when } \varepsilon = 1$$

and

$$F\left(\left(\wedge^2 V\right)_{\pm}\right) = \left(\vee^2 F(V)\right)_{\pm} \text{ (resp. } F\left(\left(\vee^2 V\right)_{\pm}\right) = \left(\wedge^2 F(V)\right)_{\pm}\text{) when } \varepsilon = -1.$$

Since $i = 2$, we have $1 = i - 1$ and it follows that $\psi^{V,?}_{i,1} = \psi^{V,?}_{g-i,1} = \psi^{V,?}_{i,g-i-1} = \psi^{V,?}_{g-i,i-1}$ and $\overline{\psi}^{V,?}_{g-i,g-i-1} = \overline{\psi}^{V,?}_{i,i-1} = \overline{\psi}^{V,?}_{g-i,1} = \overline{\psi}^{V,?}_{i,1}$. Hence, our discussion on Dirac operators is confined to the two pairings $\psi^{V,?} := \psi^{V,?}_{i,1}$ and $\overline{\psi}^{V,?}_{g-i,g-i-1} := \overline{\psi}^{V,?}_{i,1}$. Together with the multiplication maps they induce

$$\begin{array}{llll} \Delta^{\mathrm{Sym}^n(\wedge^2 V)} := \Delta^n_{i_{\wedge^g V}\circ\varphi_{2,2},a} & \mathrm{Sym}^n\left(\wedge^2 V\right) & \to & \mathrm{Sym}^{n-2}\left(\wedge^2 V\right)\otimes L_a, \\ \overline{\partial}^{\mathrm{Sym}^n(\wedge^2 V)} := \partial^n_{\overline{\psi}^{V,a},s} : & \mathrm{Sym}^n\left(\wedge^2 V\right)\otimes V^\vee & \to & \mathrm{Sym}^{n-1}(V)\otimes V, \\ \partial^{\mathrm{Sym}^n(\wedge^2 V)} := \partial^n_{\psi^{V,a},s} : & \mathrm{Sym}^n\left(\wedge^2 V\right)\otimes V & \to & \mathrm{Sym}^{n-1}(V)\otimes V^\vee\otimes L_a \end{array}$$

and their analogous where $\wedge^2 V$ is replaced by $\vee^2 V$ in the notation and the sources and the targets. On the other hand, let $\psi^{V,?}_{-} := \psi^{V,?}\circ(i_-\otimes 1_V)$, $\overline{\psi}^{V,?}_{-} := \overline{\psi}^{V,?}\circ(i_-\otimes 1_{V^\vee})$ and $\varphi_{2,2,-} := \varphi_{2,2}\circ(i_-\otimes i_-)$ be the restrictions of these pairings, where i_- is the injection associated to e_-. Then we have operators $\Delta^?_-$, $\overline{\partial}^?_-$ and $\partial^?_-$ with $? \in \left\{\mathrm{Alt}^n\left(\wedge^2 V\right), \mathrm{Sym}^n\left(\wedge^2 V\right), \mathrm{Alt}^n\left(\vee^2 V\right), \mathrm{Sym}^n\left(\vee^2 V\right)\right\}$ induced by these pairings.

Corollary 6.4. *Suppose that (V, θ) is an alternating (resp. symmetric) quaternionic object in $\mathcal{C}$ such that $L_a \simeq \mathbb{L}^{\otimes 2}$ (resp. $L_s \simeq \mathbb{L}^{\otimes 2}$) for some invertible object $\mathbb{L}$ and that we have $r_{\wedge^2 V} > 0$ (resp. $r_{\vee^2 V} > 0$)[5]. Then the Laplace and the Dirac operators induced by these $\varphi_{2,2,-}$, $\psi^{V,?}_-$ and $\overline{\psi}^{V,?}_-$ satisfies the conclusion of Theorem 4.4 (resp. Theorem 5.3).*

Furthermore, if we are given a (covariant) additive AU tensor functor $F : \mathcal{C} \to \mathcal{D}$ as above, then the conclusion of Proposition 6.2 holds true with the Laplace and the Dirac operators induced by these $\varphi_{2,2,-}$, $\psi^{V,?}_-$ and $\overline{\psi}^{V,?}_-$.

Proof. Suppose that we are given $\psi : X\otimes Y \to Z$, $\psi' : X'\otimes Y' \to Z'$ and morphisms $f : X \to X'$, $g : Y \to Y'$ and $h : Z \to Z'$ such that $\psi'\circ(f\otimes g) = h\circ\psi$. Then it is clear from §3 that we have $\partial^n_{\psi',a}\circ((\wedge^n f)\otimes g) = \left(\left(\wedge^{n-1} f\right)\otimes h\right)\circ\partial^n_{\psi,a}$ and $\partial^n_{\psi',s}\circ((\vee^n f)\otimes g) = \left(\left(\vee^{n-1} f\right)\otimes h\right)\circ\partial^n_{\psi,s}$. Similarly, when $f = g$, we deduce $\Delta^n_{\psi',a}\circ(\wedge^n f) = \left(\left(\wedge^{n-2} f\right)\otimes h\right)\circ\Delta^n_{\psi,a}$ and $\Delta^n_{\psi',s}\circ((\vee^n f)) = \left(\left(\vee^{n-2} f\right)\otimes h\right)\circ\Delta^n_{\psi,s}$. We write $\psi \to_{(f,g,h)} \psi'$ in this case. Then, setting $A_g := \wedge^g V$ or $\vee^g V$, we have by definition $i_{A_g}\circ\varphi_{2,2,-} \to_{\left(i_-,i_-,1_{L_?}\right)} i_{A_g}\circ\varphi_{2,2}$, $\psi^{V,?}_- \to_{\left(i_-,1_V,1_{L_?}\right)} \psi^{V,?}$ and $\overline{\psi}^{V,?}_- \to_{\left(i_-,1_{V^\vee},1_{L_?}\right)} \overline{\psi}^{V,?}$. It follows that, with ψ one of these pairings and ψ_- the corresponding pairing obtained by restriction, the induced Laplace or Dirac operators commutes with the canonical injections induced by $\wedge^k i_-$ (resp. $\vee^k i_-$) in the sources and the targets. We simply write Δ^n, $\overline{\partial}^n$ and ∂^n and Δ^n_-, $\overline{\partial}^n_-$ and ∂^n_- for one of these operators. Hence, if (V, θ) is alternating (resp. symmetric), we may apply Theorem 4.4 (resp. Theorem 5.3) to deduce that $\overline{\partial}^{n-1}_-\circ\partial^n_-$ or $\partial^{n-1}_-\circ\overline{\partial}^n_-$ equals $\Delta^n_-\otimes 1_V$ or $\Delta^n_-\otimes 1_{V^\vee}$ up to the isomorphism provided by this theorem. Let e be the idempotent which gives the kernel of one of the Laplace or Dirac operators Δ^n, $\overline{\partial}^n$ and ∂^n; on the other hand we have an idempotent e' of the form $e' := \mathrm{Alt}^n(e_-)\otimes 1_Z$ or $\mathrm{Sym}^n(e_-)\otimes 1_Z$ which corresponds to the injections induced by $\wedge^k i_-$ (resp. $\vee^k i_-$) in the sources of these operators Δ^n_-, $\overline{\partial}^n_-$ and ∂^n_-. Then ee' gives the kernel of the Laplace or Dirac operators Δ^n_-, $\overline{\partial}^n_-$ and ∂^n_- (once again because the operators induced by ψ or ψ_- are related by a commutative diagram involving injections). Hence the analogue of Theorem 4.4 (resp. Theorem 5.3) (3) is true. Finally, the statement about F follows from Lemma 6.3 and Proposition 6.2. □

[5]As explained in the introduction, this latter condition on the rank of the 2-powers is always fulfilled when V is Kimura positive (resp. negative).

Definition 6.5. *If we are given an alternating (resp. symmetric) quaternionic object in $\mathcal{C}$, we define* $M_2(V,\theta) := \left(\wedge^2 V\right)_-$ *(resp.* $M_2(V,\theta) := \left(\vee^2 V\right)_-$*),*

$$M_{2n}(V,\theta) := \ker\left(\Delta_-^{\mathrm{Sym}^n(\wedge^2 V)}\right) \subset \mathrm{Sym}^n\left(\left(\wedge^2 V\right)_-\right), \text{ where } n \geq 2$$

$$\text{(resp. } M_{2n}(V,\theta) := \ker\left(\Delta_-^{\mathrm{Sym}^n(\vee^2 V)}\right) \subset \mathrm{Sym}^n\left(\left(\vee^2 V\right)_-\right)\text{)}$$

and $M_1(V,\theta) := V$ *(resp.* $M_1(V,\theta) := V$*)*

$$M_{2n+1}(V,\theta) := \ker\left(\partial_-^{\mathrm{Sym}^n(\wedge^2 V)}\right) \subset \mathrm{Sym}^n\left(\left(\wedge^2 V\right)_-\right) \otimes V, \text{ where } n \geq 1$$

$$\text{(resp. } M_{2n+1}(V,\theta) := \ker\left(\partial_-^{\mathrm{Sym}^n(\vee^2 V)}\right) \subset \mathrm{Sym}^n\left(\left(\vee^2 V\right)_-\right) \otimes V\text{)}$$

It follows from Lemma 6.3 and Corollary 6.4 that these objects are canonical in the category of quaternionic objects and that, if we are given a (covariant) AU tensor functor $F : \mathcal{C} \to \mathcal{D}$ as above, then $F(M_{2n}(V,\theta)) = M_{2n}(F(V), F(\theta))$ and $F\left(M_{2n+1}^{(\oplus 2)}(V,\theta)\right) = M_{2n+1}^{(\oplus 2)}(F(V), F(\theta))$.

7. Realizations

7.1. **The quaternionic Poincaré upper half plane.** We write

$$\mathcal{P} := \mathbf{P}^1_{\mathbb{C}} - \mathbf{P}^1_{\mathbb{R}} := \mathcal{H}^+ \sqcup \mathcal{H}^- \text{ and } \mathcal{H} := \mathcal{H}^+,$$

where $\mathcal{H}^{\pm}$ is the connected component of $\mathcal{P}$ such that $\pm i \in \mathcal{H}^{\pm}$. We recall that $\mathcal{P}$ has a natural moduli interpretation in the category of analytic spaces[6] as follows. Set $L_1 := \mathbb{Z}^2$, $P_1 := \left(\mathbb{Z}^2\right)^{\vee}$ (the $\mathbb{Z}$-dual) and, for a positive integer k, $L_k := S^k_{\mathbb{Z}}(L_1)$ (the k-symmetric power of L_1) and $P_k := S^k_{\mathbb{Z}}(P_1) = L_k^{\vee}$, the space of homogeneous polynomials of degree k in two variables (X,Y). Then we have $\mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(k)\left(\mathbf{P}^1_{\mathbb{C}}\right) = P_{k,\mathbb{C}}$[7]. To give an $\mathcal{S}$-point $x : \mathcal{S} \to \mathbf{P}^1_{\mathbb{C}}$ from an analytic space $\mathcal{S}$ is to give an epimorphism $\mathcal{O}_{\mathcal{S}}(P_1) \twoheadrightarrow \mathcal{L}$[8] up to isomorphism, where $\mathcal{L}$ is an invertible sheaf on $\mathcal{S}$ and, taking $x = 1_{\mathbf{P}^1_{\mathbb{C}}}$, gives the universal quotient

$$\mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(P_1) \twoheadrightarrow \mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(1)$$

mapping the global sections $1 \otimes X, 1 \otimes Y \in \mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(P_{1,\mathbb{C}})\left(\mathbf{P}^1_{\mathbb{C}}\right)$ respectively to the global sections $X, Y \in \mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(1)\left(\mathbf{P}^1_{\mathbb{C}}\right)$. Taking duals we see that to give $x : \mathcal{S} \to \mathbf{P}^1_{\mathbb{C}}$ is the same as to give a monomorphism $\mathcal{L} \hookrightarrow \mathcal{O}_{\mathcal{S}}(L_{1,\mathbb{C}})$ up to isomorphism, where $\mathcal{L}$ is an invertible sheaf on $\mathcal{S}$ and the cokernel of the inclusion is locally free too; taking $x = 1_{\mathbf{P}^1_{\mathbb{C}}}$ gives the universal object

$$\mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(-1) \hookrightarrow \mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(L_1).$$

Although not needed, let us remark that we have indeed $\mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(L_1) = \mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(P_1)$ and the above universal epimorphism and monomorphism are part of usual canonical short exact sequence. It follows that, setting $F_x^{-1}\left(\mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(L_1)\right) := \mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(L_1)$ and $F_x^{0}\left(\mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(L_1)\right) := \mathrm{Im}\left(\mathcal{L} \hookrightarrow \mathcal{O}_{\mathcal{S}}(L_1)\right)$, the space $\mathbf{P}^1_{\mathbb{C}}$ classifies all the possible filtrations on $\mathcal{O}_{\mathcal{S}}(L_1)$ by an invertible $\mathcal{O}_{\mathcal{S}}$-module having locally free cokernel.

It is easy to realize that a necessary and sufficient condition for a point $x : \mathcal{S} \to \mathbf{P}^1_{\mathbb{C}}$ to factor through $\mathcal{P}$ is that the filtration $F_x^{\cdot}\left(\mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(L_1)\right)$ on $\mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(L_1) = \mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(L_{1,\mathbb{R}})$ gives $L_{1,\mathbb{R},\mathcal{P}}$ the structure of a variation of Hodge structures of type $\{(-1,0),(0,-1)\}$. Hence $\mathcal{P}$ classifies variations of Hodge structures on $\mathcal{S}$ of Hodge type $\{(-1,0),(0,-1)\}$ with fibers in the constant sheaf $L_{1,\mathbb{R},\mathcal{P}}$. The universal object is

$$\mathcal{L}_1 := \left(L_{1,\mathbb{R}}, \mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(-1)_{|\mathcal{P}} \hookrightarrow \mathcal{O}_{\mathbf{P}^1_{\mathbb{C}}}(L_1)_{|\mathcal{P}}\right).$$

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[6]See [GR, I.1.5] for a treatment of analytic spaces, but beware that they are called "complex spaces" in this book.

[7]If M is an A-module over a ring $A \subset B$, we write $M_B := B \otimes_A M$.

[8]If M is an A-module over a ring $A \subset \mathbb{C}$ and $\mathcal{S}$ is an analytic space, we write $M_{\mathcal{S}}$ for the associated sheaf of locally constant M-valued functions on $\mathcal{S}$ and $\mathcal{O}_{\mathcal{S}}(M) := \mathcal{O}_{\mathcal{S}} \otimes_{A_{\mathcal{S}}} M_{\mathcal{S}}$.

Let us fix B, an indefinite quaternion algebra, an identification $B_\infty \simeq \mathbf{M}_2(\mathbb{R})$ and a lattice $I \subset B$ with right (resp. left) order $R(I)$ (resp. $E(I)$). Then $I, R(I) \subset \mathbf{M}_2(\mathbb{R})$, $\mathbb{R} \otimes_{\mathbb{Z}} R(I) \simeq \mathbf{M}_2(\mathbb{R})$ are identified as $\mathbb{R}$-algebras, $\mathbb{R} \otimes_{\mathbb{Z}} I \simeq \mathbf{M}_2(\mathbb{R})$ as right $\mathbb{R} \otimes_{\mathbb{Z}} R(I) \simeq \mathbf{M}_2(\mathbb{R})$-modules and we also have $\mathcal{O}_{\mathcal{S}}(I) \simeq \mathcal{O}_{\mathcal{S}}(\mathbf{M}_2(\mathbb{R}))$ for every analytic space $\mathcal{S}$. We will mainly regard $E(I)$ as the endomorphism group of I as a right $R(I)$-module[9]. We will always with R to denote a maximal order.

Definition 7.1. *A quaternionic variation of Hodge structures on $\mathcal{S}$ ($qVHS$ in short) is a variation of Hodge structures of type $\{(-1,0),(0,-1)\}$ with fibers in the constant coherent sheaf $\mathbf{M}_2(\mathbb{R})_{\mathcal{S}}$ such that the action of $\mathbf{M}_2(\mathbb{R})$ induced on $\mathcal{O}_{\mathcal{S}}(\mathbf{M}_2(\mathbb{R}))$ by right multiplication preserves the filtration $F^{\cdot}(\mathcal{O}_{\mathcal{S}}(\mathbf{M}_2(\mathbb{R})))$ on $\mathcal{O}_{\mathcal{S}}(\mathbf{M}_2(\mathbb{R}))$.*

An I-rigidified quaternionic variation of Hodge structures on $\mathcal{S}$ ($IqVHS$ in short) is a variation of Hodge structures with fibers in the sheaf $I_{\mathcal{S}}$ such that $(\mathbb{R} \otimes_{\mathbb{Z}} I)_{\mathcal{S}} \simeq \mathbf{M}_2(\mathbb{R})_{\mathcal{S}}$ gives by transport to the right hand side the structure of a quaternionic variation of Hodge structures on $\mathcal{S}$.

We note that, having fixed $I \subset \mathbf{M}_2(\mathbb{R})$, to give a $qVHS$ is the same thing as to give an $IqVHS$.

As above, let $I \subset B$ be a lattice and let $O \subset B$ be any order.

Definition 7.2. *A fake (analytic) O-elliptic curve over $\mathcal{S}$ ($f_O EC$ in short) is $(\mathcal{A}/\mathcal{S}, i)$ where $\mathcal{A}/\mathcal{S}$ is an analytic abelian surface over $\mathcal{S}$[10] and $i : O \to End_{\mathcal{S}}(\mathcal{A})$ is a ring morphism (acting from the right on $\mathcal{A}/\mathcal{S}$).*

An I-rigidified fake (analytic) elliptic curve over $\mathcal{S}$ ($IfEC$) is a $(\mathcal{A}/\mathcal{S}, i, \rho)$ where $(\mathcal{A}/\mathcal{S}, i)$ is an $R(I)$-fake (analytic) elliptic curve over $\mathcal{S}$ and $\rho : R^1\pi_*\mathbb{Z}_{\mathcal{A}}^{\vee} \xrightarrow{\sim} I_{\mathcal{S}}$ is an isomorphism as right sheaves of $R(I)$-modules (the action on the left hand side is by functoriality).

Remark 7.3. *Although not needed in the sequel, we remark that the proof of* [BC, Ch. III, Proposition (1.5)] *applies in this analytic setting, implying that a fake (analytic) R-elliptic curve $\mathcal{A}/\mathcal{S}$ always has a canonical principal polarization (see loc.cit. for the precise conditions making the polarization canonical). In particular, it is algebrizable when $\mathcal{S}$ is algebrizable.*

Also, we remark that, if $I \simeq \mathbb{Z}^4$ is an R-module (left or right), then $I \simeq R$: indeed, $\mathbb{Q} \otimes_{\mathbb{Z}} I \simeq B$ because B is simple, so that we may view $I \subset B$ as an R-module; but the class number of B is one (by strong approximation), implying $I \simeq R$. For a fake (analytic) R-elliptic curve $(\mathcal{A}/\mathcal{S}, i)$, we have $\left(R^1\pi_*\mathbb{Z}_{\mathcal{A}}^{\vee}\right)_s \simeq \mathbb{Z}^4$ for every $s \in \mathcal{S}$ by the (topological) proper base change theorem. Because the left hand side is naturally a right R-module, we see that $R^1\pi_*\mathbb{Z}_{\mathcal{A}}^{\vee} \simeq R_{\mathcal{S}}$ when $\mathcal{S}$ is simply connected.

If we are given an I-rigidified fake (analytic) elliptic curve $(\pi : \mathcal{A} \to \mathcal{S}, i, \rho)$ over $\mathcal{S}$, the exponential map gives an exact sequence of sheaves on $\mathcal{S}$,

$$0 \to R^1\pi_*\mathbb{Z}_{\mathcal{A}}^{\vee} \to T_{\mathcal{A}/\mathcal{S}} \to \mathcal{A} \to 0. \tag{55}$$

Then we may define $F^0\left(\mathcal{O}_{\mathcal{S}}\left(R^1\pi_*\mathbb{Z}_{\mathcal{A}}^{\vee}\right)\right)$ by means of the exact sequence

$$0 \to F^0\left(\mathcal{O}_{\mathcal{S}}\left(R^1\pi_*\mathbb{Z}_{\mathcal{A}}^{\vee}\right)\right) \to \mathcal{O}_{\mathcal{S}}\left(R^1\pi_*\mathbb{Z}_{\mathcal{A}}^{\vee}\right) \to T_{\mathcal{A}/\mathcal{S}} \to 0. \tag{56}$$

By means of i the ring $R(I)$ acts on this sequence (say from the right). The rigidification yields $\rho : R^1\pi_*\mathbb{Z}_{\mathcal{A}}^{\vee} \simeq I_{\mathcal{S}}$ compatible with the right action of $R(I)$ on $I_{\mathcal{S}}$. It follows that $F^0\left(\mathcal{O}_{\mathcal{S}}\left(R^1\pi_*\mathbb{Z}_{\mathcal{A}}^{\vee}\right)\right) \simeq F^0(\mathcal{O}_{\mathcal{S}}(I)) \simeq F^0(\mathcal{O}_{\mathcal{S}}(\mathbf{M}_2(\mathbb{R})))$ (the right hand sides defined by transport) gives $I_{\mathcal{S}}$ a rigidified quaternionic variation of Hodge structures on $\mathcal{S}$ that we denote $R^1\pi_*\mathbb{Z}_{\mathcal{A}}^{\vee}$. The correspondence is indeed an equivalence of categories: when $\mathcal{S} = S^{an}$ for a complex algebraic variety S, this is an application of [Mi2, Theorem 7.13] or [De2, 4.4.3]; the reference [De1, Proposition (2.2) (ii)] contains (without proof) the analogous statement in the case of analytic elliptic curves $\mathcal{E}$ over analytic spaces and our case $B = \mathbf{M}_2(\mathbb{Q})$ and $R(I) = \mathbf{M}_2(\mathbb{Z})$ follows writing $\mathcal{A} = \mathcal{E}^2$ and splitting the rigidified quaternionic variation of Hodge structures in a similar way. The above

[9] An endomorphism φ of I as a right $R(I)$-module induces and endomorphism of $W = B$ as a right B-module. Recalling the identification $f : B \otimes B \to End_{\mathbb{Q}}(W)$ obtained when discussing quaternionic objects (defined by $b_1 \otimes b_2 \cdot x := b_1 x b_2^{\iota}$), it is easy to see that φ is induced by left multiplication by some element of B, which then belongs to $E(I)$.

[10] By analytic abelian surface over $\mathcal{S}$ we mean a proper and flat morphism $\pi : \mathcal{A} \to \mathcal{S}$ of analytic spaces of relative dimension 2, with a section $e : \mathcal{S} \to \mathcal{A}$ and a morphism $\mathcal{A} \times_{\mathcal{S}} \mathcal{A} \to \mathcal{A}$ satisfying the usual group constraints making the morphism e a unit section.

general result is shown in the proof of Proposition 7.4 below (and a posteriori equivalent to it). See also [Sh1, Theorem 3] for related results.

The following result yields a quaternionic moduli description of $\mathcal{P}$, which depends on the fixed identification $B_\infty \simeq \mathbf{M}_2(\mathbb{R})$ and the choice of a lattice $I \subset B$.

Proposition 7.4. *The analytic space $\mathcal{P}$ classifies I-rigidified fake (analytic) elliptic curve over $\mathcal{S}$. If $(\pi_I : \mathcal{A}_I \to \mathcal{P}, i_I, \rho_I)$ is the universal fake elliptic curve, we have that $R^1\pi_{I*}\mathbb{Z}^\vee_{\mathcal{A}_I} = I_{\mathcal{P}}$ and the associated variation of Hodge structures on $\mathcal{O}_{\mathcal{S}}(I) \simeq \mathcal{O}_{\mathcal{S}}(\mathbf{M}_2(\mathbb{R}))$ is given by $\mathcal{L}_1 \oplus \mathcal{L}_1$.*

Proof. Let us remark that, if we may apply [Mi2, Theorem 7.13] or [De2, 4.4.3], the above discussion would imply that we have just to classify I-rigidified quaternionic variation of Hodge structures, rather than I-rigidified fake (analytic) elliptic curves. Rather, we begin in the opposite direction.

Step 1: the analytic space $\mathcal{P}$ classifies $IqVHS$, with universal object as described above. As remarked above, the choice $B_\infty \simeq \mathbf{M}_2(\mathbb{R})$ determines $I \subset \mathbf{M}_2(\mathbb{R})$ identifying the data of $IqVHS$ and $qVHS$. Hence we classify $qVHS$. But it is easy to see that the association $\mathcal{L} \mapsto \mathcal{L} \oplus \mathcal{L}$ realizes an identification between variations of Hodge structures on $\mathcal{S}$ of Hodge type $\{(-1,0),(0,-1)\}$ with fibers in the constant coherent sheaf $L_{1,\mathbb{R}}$ and $qVHS$. The claim follows from the above description of $\mathcal{P}$ and its universal object.

Step 2: there is an $IfEC$ with prescribed associated $IqVHS$. The proof will be given in §7.5 point (2) below.

Step 3: the analytic space $\mathcal{P}$ classifies rigidified fake (analytic) elliptic curves. First, we remark that the association $(\pi : \mathcal{A} \to \mathcal{S}) \rightsquigarrow R^1\pi_*\mathbb{Z}^\vee_{\mathcal{A}}$ is fully faithful thanks to the above exact sequences (55) and (56), as remarked in the proof of [Mi2, Theorem 7.13]. As explained above, the added data i (resp. ρ) corresponds to giving the structure of an $R(I)$-module object (resp. an I-rigidification). It follows that $(\pi : \mathcal{A} \to \mathcal{S}, i, \rho) \rightsquigarrow R^1\pi_*\mathbb{Z}^\vee_{\mathcal{A}}$ is fully faithful, valued in $IqVHS$ on $\mathcal{S}$. Let us now show that this latter functor is essentially surjective; in view of Step 1, this is equivalent to our claim (and the proof will directly show the versal part of the statement about $\mathcal{P}$). Hence, suppose we are given $\mathbf{x}$, an $IqVHS$ on $\mathcal{S}$; thanks to Step 1, it gives rise to a morphism of analytic spaces $x : \mathcal{S} \to \mathcal{P}$ and we have $\mathbf{x} = x^*\mathbf{u}$, where $\mathbf{u}$ is the universal $IqVHS$ on $\mathcal{P}$ and x^* denotes the pull-back of variations of Hodge structures. According to Step 2, we have $\mathbf{u} = R^1\pi_*\mathbb{Z}^\vee_{\mathcal{A}}$. Let us remark that, by the (topological) proper base change theorem, we have $x^*R^1\pi_*\mathbb{Z}^\vee_{\mathcal{A}} = R^1\pi_*\mathbb{Z}^\vee_{\mathcal{A}\times_{\mathcal{P}}\mathcal{S}}$ as variations of Hodge structures, i.e. base change commutes with the formation of the variation of Hodge structure $R^1\pi_*\mathbb{Z}^\vee_{\mathcal{A}}$: indeed, the formation of the tangent spaces $T_{\mathcal{A}/\mathcal{S}}$ commutes with base changes, as it follows from (55) and the fact that the formation of $R^1\pi_*\mathbb{Z}^\vee_{\mathcal{A}}$ and $\mathcal{A}$ commute with base changes; then, because the formation of $T_{\mathcal{A}/\mathcal{S}}$ and $\mathcal{O}_{\mathcal{S}}\left(R^1\pi_*\mathbb{Z}^\vee_{\mathcal{A}}\right)$ commute with base changes (the latter by definition of pull-back of variations of Hodge structures, once again because the formation of $R^1\pi_*\mathbb{Z}^\vee_{\mathcal{A}}$ commutes with base changes), we see that the formation of the whole (56) commutes with base changes. Summarizing, $\mathbf{x} = R^1\pi_*\mathbb{Z}^\vee_{\mathcal{A}\times_{\mathcal{P}}\mathcal{S}}$ as variations of Hodge structures, as wanted. □

Remark 7.5. *Having fixed $B_\infty \simeq \mathbf{M}_2(\mathbb{R})$, Proposition 7.4 implies that we have that $R^1\pi_{I*}\mathbb{Q}^\vee_{\mathcal{A}_I} = B_{\mathcal{P}}$ with associated variation of Hodge structures on $\mathcal{O}_{\mathcal{S}}(B) \simeq \mathcal{O}_{\mathcal{S}}(\mathbf{M}_2(\mathbb{R}))$ given by $\mathcal{L}_1 \oplus \mathcal{L}_1$. Hence the underlying rational variation of Hodge structures $R^1\pi_*\mathbb{Q}^\vee_{\mathcal{A}} := R^1\pi_{I*}\mathbb{Q}^\vee_{\mathcal{A}_I}$ does not depend on the choice of the lattice $I \subset B$: it is endowed with a natural left B-action, extending the left $E(I)$-action and acting on the quaternionic structure (given by right multiplication).*

7.2. **Linear algebra in the category of $B^\times$-representations.** We write $x \mapsto x^\iota$ to denote the main involution of B, so that $x + x^\iota = \mathrm{Tr}(x)$ and $xx^\iota = \mathrm{Nr}(x)$. We let $B^\times$ acts on B by left multiplication, while we write B^ι to denote B on which $B^\times$ acts from the left by the rule $b \cdot x := bxb^\iota$. We write $B^0 := \ker(\mathrm{Tr})$ to denote the trace zero elements, viewed as a $B^\times$-subrepresentation of B^ι (indeed $\mathrm{Tr}(bxb^\iota) = \mathrm{Nr}(b)\mathrm{Tr}(x)$). If $V \in \mathrm{Rep}(B^\times)$ and $r \in \mathbb{Z}$, we let $V(r)$ be V on which $B^\times$ acts by $b \cdot_r v = \mathrm{Nr}^{-r}(b)bv$, so that $V(r) = V \otimes \mathbb{Q}(r)$ (canonically). We let $B^\times_+ \subset B^\times$ be the subgroup of elements having positive norm.

In [JL] certain Laplace and Dirac operators has been defined with source and target those of the subsequent Lemma 7.6. While their definition is completely explicit, it is only the definition of the Laplace operator that readily generalizes to arbitrary tensor categories; on the other hand, the definition of the Dirac operator requires the theory we have developed in order to provide good models for their kernels which have a general meaning for tensor categories. Indeed, we have the following key remark, that allows us to replace the Jordan-Livné models with ours, whose proof is left to the reader.

Lemma 7.6. *Let*

$$f_n : \mathrm{Sym}^n(B_0) \to \mathrm{Sym}^{n-2}(B_0)(-2) \ \text{ and } g_n : \mathrm{Sym}^n(B_0) \otimes B \to \mathrm{Sym}^{n-1}(B_0) \otimes B(-1)$$

be any epimorphism in $\mathrm{Rep}_{\mathbb{Q}}(B^\times)$. *Once we fix* $B \otimes \mathbb{F} \simeq \mathbf{M}_2(\mathbb{F})$, *where* $\mathbb{F}$ *is any splitting field of* B, *there are canonical isomorphisms*

$$\ker(f_n) \otimes \mathbb{F} \simeq L_{2n} \otimes \mathbb{F}, \ \ker(g_n) \otimes \mathbb{F} \simeq L^2_{2n+1} \otimes \mathbb{F}$$

which are compatible with the $(B \otimes \mathbb{F})^\times$*-action on the left hand side, the* $\mathbf{GL}_2(\mathbb{F})$*-action on the right side and the induced identification* $(B \otimes \mathbb{F})^\times \simeq \mathbf{GL}_2(\mathbb{F})$.

The following Lemma will be useful. Recall that, if M is an object in a pseudo abelian $\mathbb{Q}$-linear category on which B-acts, we may write $\wedge^2 M = \left(\wedge^2 M\right)_+ \oplus \left(\wedge^2 M\right)_-$ canonically, where B operates on $\left(\wedge^2 M\right)_-$ via the reduced norm. The following

Lemma 7.7. *Let* $\theta : B \overset{\sim}{\to} End_{\mathrm{R}ep(B^\times)}(B)$ *be the isomorphism provided by the right multiplication. Then we have, in* $\mathrm{Rep}_{\mathbb{Q}}(B^\times)$,

$$\wedge^2 B = \left(\wedge^2 B\right)_+ \oplus \left(\wedge^2 B\right)_- \ \text{ with } \left(\wedge^2 B\right)_+ \simeq \mathbb{Q}(-1)^3 \ \text{ and } \left(\wedge^2 B\right)_- \simeq B_0.$$

Since (B,θ) is an *alternating* quaternionic object, we may define

$$L^B_{2n} := M_{2n}(B,\theta) \ \text{(for } n \geq 1) \text{ and } L^{B(2)}_{2n+1} := M_{2n+1}(B,\theta) \ \text{(for } n \geq 0)$$

and it is a consequence of Lemmas 7.6 and 7.7 that, when $B \otimes \mathbb{F} \simeq \mathbf{M}_2(\mathbb{F})$,

$$L^B_{2n,\mathbb{F}} \simeq L_{2n,\mathbb{F}} \text{ and } L^{B(2)}_{2n+1,\mathbb{F}} \simeq L^2_{2r+1,\mathbb{F}}. \tag{57}$$

7.3. **Variations of Hodge structures attached to $B^\times$-representations.** In this subsection we define a $\mathbb{Q}$-additive *ACU* tensor functor, depending on the choice of an identification $B_\infty \simeq \mathbf{M}_2(\mathbb{R})$,

$$\mathcal{L} : \mathrm{Rep}\left(B^\times\right) \to \mathbf{VHS}_{\mathcal{P}}(\mathbb{Q}),$$

where $\mathbf{VHS}_{\mathcal{S}}(\mathbb{F})$ denotes the category of variations of Hodge structures on $\mathcal{S}$ with coefficients in the field $\mathbb{F} \subset \mathbb{R}$. The identification $B_\infty \simeq \mathbf{M}_2(\mathbb{R})$ induces $\mathrm{Rep}(B^\times_\infty) \simeq \mathrm{Rep}(\mathbf{GL}_{2,\mathbb{R}})$ and it follows from [Ha, Corollary 3.2 and its proof for uniqueness] that we may define a (unique up to equivalence) faithful and exact $\mathbb{Q}$-additive *ACU* tensor functor

$$\mathcal{L}_{\mathbb{R}} : \mathrm{Rep}\left(B^\times_\infty\right) \to \mathbf{VHS}_{\mathcal{P}}(\mathbb{R})$$

requiring $\mathcal{L}_{\mathbb{R}}(L_{1,\mathbb{R}}) := \mathcal{L}_1$. Since $\mathcal{O}_{\mathcal{P}}(V) = \mathcal{O}_{\mathcal{P}}(V_{\mathbb{R}})$ for every $V \in \mathrm{Rep}(B^\times)$, we deduce that the restriction of $\mathcal{L}_{\mathbb{R}}$ to $\mathrm{Rep}(B^\times) \to \mathrm{Rep}(B^\times_\infty)$ via $V \mapsto V_{\mathbb{R}}$ factors through $\mathbf{VHS}_{\mathcal{P}}(\mathbb{Q}) \to \mathbf{VHS}_{\mathcal{P}}(\mathbb{R})$ (again via scalar extension) and gives our $\mathcal{L}$. It follows from this description and Proposition 7.4 (see Remark 7.5) that we have

$$\mathcal{L}(B) = R^1\pi_*\mathbb{Q}^\vee_{\mathcal{A}}, \tag{58}$$

if B denotes the left $B^\times$-representation whose underlying vector space is B with the action given by left multiplication and quaternionic structure induced by the right multiplication θ.

Since $(\mathcal{L}(B), \mathcal{L}(\theta))$ is an *alternating* quaternionic object, we may define

$$\mathcal{L}^B_{2n} := M_{2n}(\mathcal{L}(B), \mathcal{L}(\theta)) \ \text{(for } n \geq 1) \text{ and } \mathcal{L}^{B(2)}_{2n+1} := M_{2n+1}(\mathcal{L}(B), \mathcal{L}(\theta)) \ \text{(for } n \geq 0).$$

If $K \subset \widehat{B}^\times$ is an open and compact subgroup, we may consider the Shimura curve

$$S_K(\mathbb{C}) := B^\times \backslash \left(\mathcal{P} \times \widehat{B}^\times\right)/K = B^\times_+ \backslash \left(\mathcal{H} \times \widehat{B}^\times\right)/K$$

where:

- $B^\times$ acts diagonally on $\mathcal{P} \times \widehat{B}^\times$ (via $B^\times \subset B^\times_\infty$ and the diagonal embedding $B^\times \subset \widehat{B}^\times$ on the second component)
- The action of K is trivial on $\mathcal{P}$ and by right multiplication on $\widehat{B}^\times$.

When $B \neq \mathbf{M}_2(\mathbb{Q})$, $X_K(\mathbb{C}) := S_K(\mathbb{C})$ is compact and otherwise we set $X_K(\mathbb{C}) := \overline{S_K(\mathbb{C})}$, compactified by "adding cusps". Then $\mathcal{L}(V)$ (for any $V \in \mathrm{Rep}(B^\times)$) descend to a variation of Hodge structures $\mathcal{L}_K(V)$ on $S_K(\mathbb{C})$[11].

Setting $\pi_0(S_K(\mathbb{C})) := B^\times \backslash \widehat{B}^\times / K$ we have

$$\pi_0 : S_K(\mathbb{C}) \twoheadrightarrow \pi_0(S_K(\mathbb{C})) \text{ and } \pi_K : \widehat{B}^\times \to \pi_0(S_K(\mathbb{C}))$$

If $x \in \widehat{B}^\times$ define $\Gamma_K(x) := xKx^{-1} \cap B^\times$ (resp. $\Gamma_K(x)_+ := xKx^{-1} \cap B_+^\times$), where $B^\times \subset \widehat{B}^\times$ is diagonally embedded, that we view as a subgroup $\Gamma_K(x) \subset B^\times \subset B_\infty^\times = \mathbf{GL}_2(\mathbb{R})$. We have the mutually inverse bijections

$$p_x : \pi_0^{-1}(\pi_K(x)) = B^\times \backslash B^\times (\mathcal{P} \times xK/K) \overset{\sim}{\to} \Gamma_K(x) \backslash \mathcal{P} \text{ and } \iota_x : \Gamma_K(x) \backslash \mathcal{P} \overset{\sim}{\to} \pi_0^{-1}(\pi_K(x))$$

defined by the rules

$$[\tau, xk] \mapsto [\tau] \text{ and } [\tau] \mapsto [\tau, x]$$

Then

$$S_K(\mathbb{C}) = \coprod_{\pi_K(x) \in \pi_0(X_K(\mathbb{C}))} \pi_0^{-1}(\pi_K(x)) \simeq \coprod_{\pi_K(x) \in \pi_0(X_K(\mathbb{C}))} \Gamma_K(x) \backslash \mathcal{P} \text{ and } \Gamma_K(x)_+ \backslash \mathcal{H} \overset{\sim}{\to} \Gamma_K(x) \backslash \mathcal{P} \tag{59}$$

It follows from the Eichler-Shimura isomorphism (see [Hi, Ch. 6] and [GSS, §3.2] or [RS, §2.4] for the statement in the quaternionic setting) and (57) that the cohomology groups (let $(?) = \phi$ when k is even and $(?) = (2)$ when k is odd)

$$H^1\left(S_K(\mathbb{C}), \mathcal{L}_{k,K}^{B(?)}\right) \simeq \bigoplus_{\pi_K(x) \in \pi_0(X_K(\mathbb{C}))} H^1\left(\Gamma_K(x), L_k^{B(?)}\right), \tag{60}$$

afford weight $k+2$ modular forms of level K when k is *even* and two copies of them when k is *odd*. Indeed, it is not difficult to define Hecke operators on the family $\left\{\mathcal{L}_{k,K}^{B(?)}\right\}_K$ by means of correspondences, which are given by double cosets on the right hand side; we also remark that the left hand side has a natural Hodge structure endowed with Hecke multiplication. We remark that, when $B \neq \mathbf{M}_2(\mathbb{Q})$ the Hecke action on (60) is purely cuspidal, whereas in case $B = \mathbf{M}_2(\mathbb{Q})$ the Hecke action factors (60) as the direct sum of its cuspidal and Eisenstin part. It is a non-trivial task to lift this decomposition at the motivic level in order to single out a motive of cuspidal modular forms: this is done in [Sc] and we will not touch this problem.

7.4. **The motives of quaternionic modular forms and their realizations.** Let $K \subset \widehat{B}^\times$ be an open and compact subgroup which is small enough so that S_K is a fine moduli space and let $\pi_K : A_K \to S_K$ be the universal level K fake elliptic curve over S_K. Consider the relative motive $h(A_K)$ as an object of $\mathbf{Mot}_+^0(S_K, \mathbb{F})$, where h is the contravariant functor

$$h \colon \mathbf{Sch}(S_K) \to \mathbf{Mot}_+^0(S_K, \mathbb{F})$$

from the category of smooth and projective schemes over S to the category of Chow motives with coefficients in a field $\mathbb{F}$. By functoriality of the motivic decomposition, there is $\theta : B \to End\left(h^1(A_K)\right)$ making $\left(h^1(A_K), \theta\right)$ a *symmetric* quaternionic object, and we may define

$$M_{2n,K}^B := M_{2n}\left(h^1(A_K), \theta\right) \text{ (for } n \geq 1\text{) and } M_{2n+1,K}^{B(2)} := M_{2n+1}\left(h^1(A_K), \theta\right) \text{ (for } n \geq 0\text{)}$$

There is a realization functor

$$R_{S_K} : \mathbf{Mot}_+^0(S_K, \mathbb{F}) \to D^b\left(\mathbf{VMHS}(S_K, \mathbb{F})\right)$$

extending the correspondence mapping $\pi : X \to S_K$ to $R\pi_* \mathbb{F}_X^\vee$. Here $\mathbf{VMHS}(S_K, \mathbb{F})$ denotes the abelian category of variations of mixed Hodge structures over S_K with coefficients in $\mathbb{F}$. See [PS, 14.4] for details.

Theorem 7.8. *Taking $F = \mathbb{Q}$ we have the following realizations.*

(1) *Suppose that $2n \geq 2$ is even. Then:*

$$R(M_{2n,K}^B) = \mathcal{L}_{2n,K}^B[-2n].$$

[11]Suffices indeed to check that $\mathcal{L}_1$ descend to $S_K(\mathbb{C})$ in order to get a functor $\mathcal{L}_{\mathbb{R},K}$ (from [Ha]) valued in $\mathbf{VHS}_{S_K(\mathbb{C})}(\mathbb{R})$ and then appeal to the uniqueness to deduce that $\mathcal{L}_{\mathbb{R},K}(V)$ is obtained from $\mathcal{L}_{\mathbb{R}}(V)$ by descend for every V. Then one can promote the restriction of $\mathcal{L}_{\mathbb{R},K}$ to $\mathrm{Rep}(B^\times)$ to take values in $\mathbf{VHS}_{S_K(\mathbb{C})}(\mathbb{Q})$, exactly as above.

(2) *Suppose that* $2n+1\geq 3$ *is odd. Then:*

$$R(M^{B(2)}_{2n+1,K})=\mathcal{L}^{B(2)}_{2n+1,K}[-(2n+1)].$$

Proof. As in [DM, Remarks 2) after Corollary 3.2] one has $R\left(h^1(A_K)\right)=R^1\pi_{K*}\mathbb{Q}^\vee_{A_K}[-1]$ and $R^1\pi_{K*}\mathbb{Q}^\vee_{A_K}$ is obtained by descending the sheaf $R^1\pi_*\mathbb{Q}^\vee_{\mathcal{A}}$ (see Proposition 7.10 below for details), so that (58) implies $R\left(h^1(A_K)\right)=\mathcal{L}_K(B)[-1]$. Since R_{S_K} is a AU tensor functor (indeed anti-commutative, see [Ku, Remark (2.6.1)]), we deduce from the remark after Definition 6.5 that (let $(?)=\phi$ when k is even and $(?)=(2)$ when k is odd)

$$R(M^{B(?)}_{k,K})=M_k\left(\mathcal{L}_K(B)[-1],\mathcal{L}_K(\theta)\right)=M_k\left(\mathcal{L}_K(B),\mathcal{L}_K(\theta)\right)[-k]=\mathcal{L}^{B(?)}_{k,K}[-k].$$

$\square$

Together with (60) and recalling that the group cohomology of L_k is concentrated in degree 1, we deduce the following result.

Corollary 7.9. *Let H be the Betti realization functor, valued in* $\mathbf{VHS}(\mathbb{Q})$*, and let view* $M^B_{2n,K}$ *and* $M^{B(2)}_{2n+1,K}$ *as motives defined over* $\mathbb{Q}$.

(1) *Suppose that* $2n\geq 2$ *is even. Then:*

$$H^i(M^B_{2n,K})=\begin{cases} H^1\left(S_K(\mathbb{C}),\mathcal{L}^{B(?)}_{k,K}\right) & \textit{if } i=2n+1\\ 0 & \textit{otherwise.}\end{cases}$$

(2) *Suppose that* $2n+1\geq 3$ *is odd. Then:*

$$H^i(M^{B(2)}_{2n+1,K})=\begin{cases} H^1\left(S_K(\mathbb{C}),\mathcal{L}^{B(?)}_{k,K}\right) & \textit{if } i=2n+2\\ 0 & \textit{otherwise.}\end{cases}$$

As explained after (60), this motivates our designation of $M^B_{2n,K}$ (resp. $M^{B(2)}_{2n+1,K}$) as the motive of (resp. two copies of) level K and weight $2n+2$ (resp. $2n+3$) modular forms: indeed the functoriality of our construction implies that the Hecke correspondences induces a Hecke multiplication on $M^B_{2n,K}$ (resp. $M^{B(2)}_{2n+1,K}$), which is compatible with that on the realizations. As remarked after (60), in case $B=\mathbf{M}_2(\mathbb{Q})$ Scholl has been able to single out a motive of cuspidal modular forms $M^{\mathrm{cusp}}_{m,K}$. The concrete construction of its motive is actually different, as we always start the game with two copies of the universal elliptic curve. Its method is finer even on the open modular curve: it gives $M_{m,K}$ such that $M_{m,K}$ has the same realization of $M^B_{2n,K}$ when $m=2n$ and realize one of the two copies of the realization of $M^{B(2)}_{2n+1,K}$ in case $m=2n+1$. The abstract approach employed here for computing the realizations, inspired by [IS], easily adapts to the other realizations: one has only to appropriately replace (58) (which is, for example, the deeper [IS, Lemma 5.10] in the p-adic realm considered there).

7.5. **Final remarks.** In this §, we collect basic facts that are surely well-known to experts about analytic families of fake elliptic curves, mainly due to Shimura, following the point of view of [De1] in the case of modular curves (see also [C, §6.1]). This will allow us to finish the proof of Proposition 7.4, as well as of Theorem 7.8 thanks to the following result, whose proof will be given at the very end of the section.

Proposition 7.10. *If K is small enough, then $R^1\pi_{K*}\mathbb{Q}^\vee_{A_K}$ is obtained by descending the sheaf $R^1\pi_*\mathbb{Q}^\vee_{\mathcal{A}}$.*

Recall our fixed identification $B_\infty\simeq\mathbf{M}_2(\mathbb{R})$ and note that, for every $\tau\in\mathcal{P}$, we may identify (see [Sh1, Prop. 14] or [P, Lemma 1.6]):

$$\Phi_\tau: B_\infty\simeq\mathbf{M}_2(\mathbb{R})\xrightarrow{\sim}\mathbb{C}^2,\text{ via } w\mapsto w^\iota\begin{pmatrix}\tau\\1\end{pmatrix},$$

thus getting a $\mathcal{C}^\infty$-morphism

$$\Phi:\mathcal{P}\times B_\infty\xrightarrow{\sim}\mathcal{P}\times\mathbb{C}^2,\text{ via } (\tau,w)\mapsto(\tau,\Phi_\tau(w)).$$

If $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we define $j(\beta,\tau) = c\tau + d$. We remark the formula

$$j(\beta^\iota,\tau)\,\Phi_{\beta^\iota\tau}(w) = \Phi_\tau(\beta w) \tag{61}$$

Suppose that we are given a left $B_\infty^\times$-representation V and let us consider $B_\infty^\times \ltimes V$, the multiplication being defined by the rule $(\beta_1,v_1)(\beta_2,v_2) := (\beta_1\beta_2, \beta_1 v_2 + v_1)$. We define a left action of $B_\infty^\times \ltimes V$ on $\mathcal{P}\times V$ by the rule

$$\begin{aligned} &\left(B_\infty^\times \ltimes V\right)\times(\mathcal{P}\times V) \to \mathcal{P}\times V \\ &(\beta,b)\cdot(\tau,w) := (\beta\tau, \beta w + b). \end{aligned} \tag{62}$$

On the other hand, we define a left action of $B_\infty^\times \ltimes B_\infty$ on $\mathcal{P}\times\mathbb{C}^2$ by the rule

$$\begin{aligned} &\left(B_\infty^\times \ltimes B_\infty\right)\times\left(\mathcal{P}\times\mathbb{C}^2\right) \to \mathcal{P}\times\mathbb{C}^2 \\ &(\beta,b)\cdot(\tau,w) := \left(\beta\tau, \frac{\det(\beta)}{j(\beta,\tau)}\left(w + b^\iota\beta^{-\iota}\begin{pmatrix}\tau \\ 1\end{pmatrix}\right)\right) \end{aligned} \tag{63}$$

It is easy to see, using (61), that Φ is $B_\infty^\times \ltimes B_\infty$-equivariant, thus making the actions (62) and (63) correspond to each other:

$$\Phi((\beta,b)\cdot(\tau,w)) = (\beta,b)\cdot\Phi((\tau,w)). \tag{64}$$

Let us now give some constructions showing the existence of a I-rigidified fake (analytic) elliptic curve as required in Step 2 of the proof of Proposition 7.4, explaining the relationship with the level K fake elliptic curves and making explicit the sheaves of locally constant functions underlying the $\mathcal{L}(V)$'s and the data obtained from the open and compact subgroups of $\widehat{B}^\times$.

(Step 1) Let $\Gamma \subset E(I)^\times$ be a subgroup acting without fixed points on $\mathcal{P}$ and consider

$$\pi_{\Gamma,I} : \mathcal{A}_{\Gamma,I} := (\Gamma \ltimes I)\backslash(\mathcal{P}\times B_\infty) \to \Gamma\backslash\mathcal{P},$$

where $\pi_{\Gamma,I}$ is induced by the first projection. This is a morphism of analytic spaces, since the fact that Γ acts without fixed points on $\mathcal{P}$ makes the induced action of $\Gamma \ltimes I$ properly discontinuous. This also implies that for any other $\Gamma' \subset \Gamma$ the following diagram is cartesian

$$\begin{array}{ccc} \mathcal{A}_{\Gamma',I} & \to & \mathcal{A}_{\Gamma,I} \\ \pi_{\Gamma',I}\downarrow & & \downarrow\pi_{\Gamma,I} \\ \Gamma'\backslash\mathcal{P} & \to & \Gamma\backslash\mathcal{P} \end{array} \quad \text{and} \quad \pi_{\Gamma,I} = \Gamma\backslash\pi_{\Gamma',I}. \tag{65}$$

Note that, by construction, the second projection $\mathcal{P}\times\mathbb{C}^2 \to \mathbb{C}^2$ induces

$$\pi_{1,I}^{-1}(\tau) \xrightarrow{\sim} \Phi_\tau(I)\backslash\mathbb{C}^2. \tag{66}$$

More generally, because (65) is cartesian, we see that (66) implies that we have a non-canonical bijection $\pi_{\Gamma,I}^{-1}(\Gamma\tau) \simeq \Phi_\tau(I)\backslash\mathbb{C}^2$. Note that, when $\Gamma = \{1\}$, we have an evident morphism $\mathcal{A}_{1,I}\times_{\mathcal{P}}\mathcal{A}_{1,I} \to \mathcal{A}_{1,I}$ and a section making $\pi_{1,I}$ an analytic abelian surface over $\mathcal{P}$. Indeed, writing $\mathcal{A}_{\Gamma,I} = \Gamma\backslash\mathcal{A}_{1,I}$, we see that $\pi_{\Gamma,I}$ has a natural structure of fake (analytic) $R(I)$-elliptic curve over $\mathcal{P}$, with right $R(I)$-multiplication $i_{\Gamma,I} : R(I) \to End_{\Gamma\backslash\mathcal{P}}(\mathcal{A}_{\Gamma,I})$ induced by $(\tau,w) \mapsto (\tau, wr)$.

(Step 2) When $\Gamma = \{1\}$, we remove the subscript Γ from the notation and the above construction yields

$$\pi_I : \mathcal{A}_I := (1\ltimes I)\backslash(\mathcal{P}\times B_\infty) \to \mathcal{P}$$

and $i_I : R(I) \to End_{\mathcal{P}}(\mathcal{A}_I)$ which is further endowed with an I-rigidification given by the identity $\rho_I : R^1\pi_*\mathbb{Z}_{\mathcal{A}}^\vee = I_{\mathcal{P}}$. It is easy to see that we have $R^1\pi_*\mathbb{Z}_{\mathcal{A}}^\vee = I_{\mathcal{P}}$ as rigidified quaternionic variations of Hodge structures, i.e. the associated variation of Hodge structures on $\mathcal{O}_{\mathcal{S}}(I) \simeq \mathcal{O}_{\mathcal{S}}(\mathbf{M}_2(\mathbb{R}))$ is $\mathcal{L}_1 \oplus \mathcal{L}_1$: indeed, suffices to look at the complex structure on the fiber over τ, which in view of (66) is obtained identifying $\mathbb{R}\otimes_{\mathbb{Q}} I \simeq \mathbf{M}_2(\mathbb{R}) \xrightarrow{\sim} \mathbb{C}^2$, via $b \mapsto b^\iota\begin{pmatrix}\tau \\ 1\end{pmatrix}$; this is two copies of the complex structure obtained identifying $\mathbb{R}^2 \xrightarrow{\sim} \mathbb{C}$ via $(x,y) \mapsto x\tau + y$, i.e. it is two copies of the fiber of $\mathcal{L}_1$ over τ. This completes the proof of Proposition 7.4 and then we know that (π_I, i_I, ρ_I) is the universal I-rigidified fake (analytic) elliptic curve.

We remark that the rule

$$\beta^\iota(\pi : \mathcal{A} \to \mathcal{S}, i, \rho) = (\pi : \mathcal{A} \to \mathcal{S}, i, \rho)\,\beta := (\pi : \mathcal{A} \to \mathcal{S}, i, \beta\circ\rho), \tag{67}$$

where $\beta \in E(I)^{\times}$ denotes the $R(I)$-linear morphism given by left multiplication by β, induces a left $E(I)^{\iota\times}$-action on $IqVHS$. Applying this definition to the universal family $(\pi_I : \mathcal{A}_I \to \mathcal{P}, i_I, \rho_I)$ gives a unique morphism $[\beta^{\iota}] : \mathcal{P} \to \mathcal{P}$ such that $[\beta^{\iota}]^{*}(\pi_I : \mathcal{A}_I \to \mathcal{P}, i_I, \rho_I) = (\pi_I : \mathcal{A}_I \to \mathcal{P}, i_I, \beta \circ \rho_I)$. The multiplication by $j(\beta^{\iota}, \tau)$ induces an isomorphism $\Phi_{\beta^{\iota}\tau}(I)\backslash\mathbb{C}^2 \simeq j(\beta^{\iota}, \tau)\,\Phi_{\beta^{\iota}\tau}(I)\backslash\mathbb{C}^2$ and we have $j(\beta^{\iota}, \tau)\,\Phi_{\beta^{\iota}\tau}(I)\backslash\mathbb{C}^2 \simeq \Phi_{\tau}(\beta I)\backslash\mathbb{C}^2 = \Phi_{\tau}(I)\backslash\mathbb{C}^2$ in view of (61); also, it follows from (61) that, under this isomorphism, the identification identity $H_1(\Phi_{\beta^{\iota}\tau}(I)\backslash\mathbb{C}^2, \mathbb{Z}) = I$ corresponds to $H_1(\Phi_{\tau}(I)\backslash\mathbb{C}^2, \mathbb{Z}) = I \xrightarrow{\beta\cdot} I$. Hence $[\beta^{\iota}](\tau) = \beta^{\iota}\tau$ and, by uniqueness, we see that (62) (and (63)) describes the resulting cartesian diagram $\mathcal{A}_I/\mathcal{P} \to \mathcal{A}_I/\mathcal{P}$. In particular, we see that

$$\pi_{\Gamma,I} = \Gamma\backslash\pi_I \text{ with } \Gamma \subset E(I)^{\times} \text{ acting via } (67) = (62). \tag{68}$$

Furthermore, if for an integer $N \geq 1$ we define$\Gamma_{I,N} := \left\{\gamma \in E(I)^{\times} : \gamma \equiv 1 \bmod IN\right\}$, then the rigidification $\rho_I : R^1\pi_{*}\mathbb{Z}_{\mathcal{A}}^{\vee} = I_{\mathcal{P}}$ yields a natural isomorphism $\rho_{I,N} : \mathcal{A}_{\Gamma_{I,N},I}[N] = I\backslash N^{-1}I$.

(Step 3) Fix a maximal order R and, for an integer $N \geq 1$, let $K_N \subset \widehat{R}^{\times}$ be the normal subgroup of elements that are congruent to 1 modulo N. Because B has class number one (by strong approximation), we have $\widehat{B}^{\times} = B^{\times}\widehat{R}^{\times}$ and we see that, for every $K \subset \widehat{R}^{\times}$, we have

$$\pi_0(S_K(\mathbb{C})) = B^{\times}\backslash B^{\times}\widehat{R}^{\times}/K \xleftarrow{\sim} R^{\times}\backslash\widehat{R}^{\times}/K \text{ and } \Gamma_K(x) \subset R^{\times} \tag{69}$$

choosing $x \in \widehat{R}^{\times}$ in the definition of $\Gamma_K(x)$ that appears in the decomposition (59). We may therefore apply the above considerations **(Step 1)** and **(Step 2)** to $\pi_{\Gamma_K(x),R} : \mathcal{A}_{\Gamma_K(x),R} \to \Gamma_K(x)\backslash\mathcal{P}$ assuming K is so small that $\Gamma_K(x)$ acts without fixed points on $\mathcal{P}$. We define, for every $K \subset \widehat{R}^{\times}$,

$$\pi_{K,R} : \mathcal{A}_{K,R} := \left(R^{\times} \ltimes R\right)\backslash\left(\mathcal{P} \times B_{\infty} \times \widehat{R}^{\times}\right)/K \to B^{\times}\backslash\left(\mathcal{P} \times \widehat{B}^{\times}\right)/K = S_K(\mathbb{C}),$$

where π_K is induced by the first and the third projection, $R^{\times} \ltimes R$ acts via $(\beta, b)\cdot(\tau, w, x) = ((\beta, b)\cdot(\tau, w), \beta x) = (\beta\tau, \beta w + b, \beta x)$ and the action of K is trivial on $\mathcal{P} \times B_{\infty}$ and by right multiplication on $\widehat{R}^{\times}$. Choosing $x \in \widehat{R}^{\times}$ as in (69), we see that we have the mutually inverse bijections

$$\begin{aligned} p_x &: \pi_{K.R}^{-1}\left(\pi_0^{-1}(\pi_K(x))\right) = \left(R^{\times} \ltimes R\right)\backslash\left(R^{\times} \ltimes R\right)(\mathcal{P} \times B_{\infty} \times xK/K) \xrightarrow{\sim} \Gamma_K(x)\backslash\mathcal{A}_R = \mathcal{A}_{\Gamma_K(x),R} \\ \iota_x &: \mathcal{A}_{\Gamma_K(x),R} = \Gamma_K(x)\backslash\mathcal{A}_R \xrightarrow{\sim} \pi_K^{-1}\left(\pi_0^{-1}(\pi_K(x))\right) \end{aligned} \tag{70}$$

defined by the rules

$$[\tau, w, xk] \mapsto [\tau, w] \text{ and } [\tau, w] \mapsto [\tau, w, x].$$

(Step 4) Again assuming that $K \subset \widehat{R}^{\times}$, let us show that $\pi_{K,R} : \mathcal{A}_{K,R} \to S_K(\mathbb{C})$ is canonically identified with $\pi_K : A_K(\mathbb{C}) \to S_K(\mathbb{C})$ for K small enough . Indeed, the general result follows from the case $K = K_N$ (because $\{K_N\}$ is cofinal, hence we may choose $K_N \subset K$ and take K-invariants from $\pi_{K_N,R} = \pi_{K_N}$ to get $\pi_{K,R} = \pi_K$). In particular, we have

$$\Gamma_{K_N}(x) := xK_Nx^{-1} \cap B^{\times} = K_N \cap B^{\times} =: \Gamma_{R,N},$$

the subgroup of the elements of $\Gamma_R := \widehat{R}^{\times} \cap B^{\times}$ that are congruent to one modulo N. As explained in [BC, Ch. III, Théorème (1.1)], S_{K_N} classifies fake R-elliptic curves $(\pi : A \to S, i)$ together with an isomorphism of right R-modules $\rho_N : A[N] \simeq \frac{N^{-1}R}{R}$: we identify $S_{K_N}(S)$ with the set of isomorphism classes of these triples and, abusively, for an analytic space $\mathcal{S}$ we write $S_{K_N}(\mathcal{S})$ for the corresponding moduli problem in the category of analytic spaces. Let us consider a subfunctor $\mathcal{S}_{K_N}^{\circ} \subset S_{K_N}$ defined as follows: it is characterized by the fact that, if $\mathcal{S}$ is connected, it classifies triples $(\pi : \mathcal{A} \to \mathcal{S}, i, \rho_N)$ with the property that, writing $\widetilde{\mathcal{S}} \to \mathcal{S}$ for the universal cover and $\left(\widetilde{\pi} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{S}}, \widetilde{i}, \widetilde{\rho_N}\right)$ for the pull-back of $(\pi : \mathcal{A} \to \mathcal{S}, i, \rho_N)$ to $\widetilde{\mathcal{S}}$, there is a rigidification $\widetilde{\rho} : R^1\widetilde{\pi}_{*}\mathbb{Z}_{\widetilde{\mathcal{A}}}^{\vee} \xrightarrow{\sim} R_{\widetilde{\mathcal{S}}}$ which lifts $\widetilde{\rho_N}$. We remark that $\left(\frac{R}{NR}\right)^{\times}$ acts simply transitively from the right on the set of isomorphism $\widetilde{\rho_N} : \widetilde{\mathcal{A}}[N] \simeq \frac{N^{-1}R}{R}$ (resp. $\rho_N : \mathcal{A}[N] \simeq \frac{N^{-1}R}{R}$) similarly as in (67) and, by Remark 7.3, there is always a rigidification $R^1\widetilde{\pi}_{*}\mathbb{Z}_{\widetilde{\mathcal{A}}}^{\vee} \xrightarrow{\sim} R_{\widetilde{\mathcal{S}}}$, which then induces $x \circ \widetilde{\rho_N}$ for some $x \in \left(\frac{R}{NR}\right)^{\times}$: in other words, $S_{K_N} = \bigcup_{x \in \left(\frac{R}{NR}\right)^{\times}} \mathcal{S}_{K_N}^{\circ}x$. Indeed, we can refine the union as follows. First, we remark that $\mathcal{S}_{K_N}^{\circ}r = \mathcal{S}_{K_N}^{\circ}$ for every r coming from $R^{\times}$ because $R^{\times}$ acts on the rigidifications via (67), implying that $S_{K_N} = \bigcup_{x \in R^{\times}\backslash\widehat{R}^{\times}/K_N} \mathcal{S}_{K_N}^{\circ}x$. Suppose that $(\pi : A \to S, i, \rho_N)$ is an $\mathcal{S}$ point in $\mathcal{S}_{K_N}^{\circ}x_1 \cap \mathcal{S}_{K_N}^{\circ}x_2$, meaning that $x_1 \circ \widetilde{\rho_N}$ and $x_2 \circ \widetilde{\rho_N}$ lift to rigidifications $\widetilde{\rho}_1, \widetilde{\rho}_2 : R^1\widetilde{\pi}_*\mathbb{Z}^\vee_{\widetilde{\mathcal{A}}} \overset{\sim}{\to} R_{\widetilde{\mathcal{S}}}$. Because the $R^\times$-action (67) on the set of rigidification is simply transitive, we may write $\widetilde{\rho}_2 = r \circ \widetilde{\rho}_1$ for some $r \in R^\times$ and then we see that $x_2 \circ \widetilde{\rho_N} = r \circ x_1 \circ \widetilde{\rho_N}$, which implies $x_2 = r \circ x_1$ in $\left(\frac{R}{NR}\right)^\times = \widehat{R}^\times / K_N$. In other words,

$$S_{K_N} = \bigsqcup_{x \in R^\times \backslash \widehat{R}^\times / K_N} \mathcal{S}^\circ_{K_N} x. \tag{71}$$

We would like to understand $\pi_{K_N} : A_{K_N}(\mathbb{C}) \to S_{K_N}(\mathbb{C})$ as a morphism of analytic spaces. Comparing (59) with (69) and (71), suffices to understand $\mathcal{S}^\circ_{K_N}$. Let us identify $\Gamma_{R,N}\backslash\mathcal{P} \simeq \mathcal{S}^\circ_{K_N}$ by showing that

$$\left(\pi_{\Gamma_{R,N},R} : \mathcal{A}_{\Gamma_{R,N},R} \to \Gamma_{R,N}\backslash\mathcal{P}, i_{\Gamma_{R,N},R}, \rho_{R,N}\right)$$

is the universal object that classifies the triples $(\pi : \mathcal{A} \to \mathcal{S}, i, \rho_N) \in \mathcal{S}^\circ_{K_N}(\mathcal{S})$ as above (see [C, Theorem 6.1.10] for the analogue of this result in the modular case and [Sh1, Theorem 3 and Proposition 15] or [P, Propositions 1.7, 1.11] for a description of the fibers of $\pi_{\Gamma_N,R}$ as a classifying space). Suppose we are given an R-fake (analytic) elliptic curve over $\mathcal{S}$ with full level N structure $(\pi : \mathcal{A} \to \mathcal{S}, i, \rho_N)$, where $\mathcal{S}$ is connected. Let $\widetilde{\mathcal{S}} \to \mathcal{S}$ be the universal cover, write $\left(\widetilde{\pi} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{S}}, \widetilde{i}, \widetilde{\rho_N}\right)$ for the pull-back of $(\pi : \mathcal{A} \to \mathcal{S}, i, \rho_N)$ to $\widetilde{\mathcal{S}}$ and choose a rigidification $\widetilde{\rho} : R^1\widetilde{\pi}_*\mathbb{Z}^\vee_{\widetilde{\mathcal{A}}} \overset{\sim}{\to} R_{\widetilde{\mathcal{S}}}$ which lifts $\widetilde{\rho_N}$: two different lifts being uniquely determined up to replacing $\widetilde{\rho}$ with $\beta \circ \widetilde{\rho}$ for some $\beta \in \Gamma_{R,N}$. We remark that the rigidification $\widetilde{\rho}$ yields a representation

$$\pi_1(\mathcal{S}) \to \mathrm{Aut}\left(R^1\widetilde{\pi}_*\mathbb{Z}^\vee_{\widetilde{\mathcal{A}}}\right) \simeq \mathrm{Aut}\left(R_{\widetilde{\mathcal{S}}}\right) = E(R)^\times = R^\times, \tag{72}$$

whose image is contained in $\Gamma_{R,N}$ because the elements of $\pi_1(\mathcal{S})$ act as the identity on $\widetilde{\rho_N} : \widetilde{\mathcal{A}}[N] \simeq \frac{N^{-1}R}{R}$ which comes from the constant sheaf $\rho_N : \mathcal{A}[N] \simeq \frac{N^{-1}R}{R}$. By Proposition 7.4, there is a unique morphism $\widetilde{x} : \widetilde{\mathcal{S}} \to \mathcal{P}$ such that $\widetilde{x}^*(\pi_R : \mathcal{A}_R \to \mathcal{P}, i_R, \rho_R) = \left(\widetilde{\pi} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{S}}, \widetilde{i}, \widetilde{\rho}\right)$, which is $\pi_1(\mathcal{S})$-equivariant with respect to (72) (because $\tilde{x}^*(\tilde{\rho}) = \rho_R$ and (67) = (62) in (68)). In particular, writing $\rho_{R,N} : \mathcal{A}_R[N] = R\backslash N^{-1}R$ for the identification induced by ρ_R (same notation already in force for $\mathcal{A}_{\Gamma_{R,N},R}$), we have $\widetilde{x}^*\left(\rho_{R,N}\right) = \widetilde{\rho_N}$. We see that we have constructed a commutative diagram

$$\begin{array}{ccc} \left(\widetilde{\pi} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{S}}, \widetilde{i}, \widetilde{\rho_N}\right) & \overset{\widetilde{x}}{\to} & \left(\pi_R : \mathcal{A}_R \to \mathcal{P}, i_R, \rho_{R,N}\right) \\ \downarrow & & \downarrow \\ (\pi : \mathcal{A} \to \mathcal{S}, i, \rho_N) & \overset{x}{\dashrightarrow} & \left(\pi_{\Gamma_{R,N},R} : \mathcal{A}_{\Gamma_{R,N},R} \to \Gamma_{R,N}\backslash\mathcal{P}, i_{\Gamma_{R,N},R}, \rho_{R,N}\right) \end{array} \tag{73}$$

in which all the arrows are cartesian (the most right because (65) is cartesian and by definition of the $\rho_{R,N}$'s) and we are looking for the dotted arrow x. First, we remark that it exists because $\pi = \pi_1(\mathcal{S})\backslash\widetilde{\pi}$, the image of $\pi_1(\mathcal{S})$ in $R^\times$ via (72) is contained in $\Gamma_{R,N}$ and $\widetilde{x}$ is $\pi_1(\mathcal{S})$-equivariant with respect to (72). It is a cartesian arrow because the left vertical arrow is both surjective on the base $\widetilde{\mathcal{S}} \to \mathcal{S}$ and cartesian and the composition of it with x is cartesian (by the commutativity of (73), because $\widetilde{x}$ and the right vertical arrow are cartesian). Because x is uniquely determined (since the left vertical arrow is surjective) and, as remarked, two lifts of $\widetilde{\rho_N}$ differs by some $\beta \in \Gamma_{R,N}$ which leaves x unchanged, also the uniqueness of x is proved.

(Step 5) If V is a left $B^\times$-representation, then we define $L(V) := \mathcal{P} \times V \to \mathcal{P}$. If $K \subset \widehat{B}^\times$ is an open and compact subgroup, we may form

$$L_K(V) := \left(B^\times \ltimes 1\right)\backslash\left(\mathcal{P} \times B_\infty \times \widehat{B}^\times\right)/K \to B^\times\backslash\left(\mathcal{P} \times \widehat{B}^\times\right)/K = S_K(\mathbb{C}).$$

We identify $L(V)$ and $L_K(V)$ with the associated sheaf of sections. It is clear that we have $\mathcal{L}(V) = L(V)$ and $\mathcal{L}_K(V) = L_K(V)$, where we abusively write $\mathcal{L}(V)$ and $\mathcal{L}_K(V)$ to denote the underlying sheaves of locally constant functions[12]. Taking $V = B$, it is clear from $\mathcal{A}_R = (1 \ltimes R)\backslash(\mathcal{P} \times B_\infty)$ that we have $L(B) = R^1\pi_*\mathbb{Q}^\vee_{\mathcal{A}}$, thus confirming (58).

Finally, the proof of Proposition 7.10 is straightforward. Given K small enough, we have $K \subset \widehat{R}^\times$ for some maximal order R. It then follows from **(Step 4)** that we have $R^1\pi_{K*}\mathbb{Q}^\vee_{A_K} = R^1\pi_{K,R*}\mathbb{Q}^\vee_{\mathcal{A}_{K,R}}$, where the right hand side is obtained descending the sheaves $R^1\pi_{R*}\mathbb{Q}^\vee_{\mathcal{A}_R}$, in view of (70) and the cartesian diagram (65). This concludes the proof.

[12]Indeed, we may also work with $B^\times_\infty$-representations, apply [Ha, Corollary 3.2] to construct $V \mapsto L(V)$ (resp. $L_K(V)$) by means of an $L_\mathbb{R}$ as we did for $\mathcal{L}$ (resp. $\mathcal{L}_K$) and the uniqueness implies that we have just to remark that $\mathcal{L}_\mathbb{R}(L_{1,\mathbb{R}}) = L_\mathbb{R}(L_{1,\mathbb{R}})$.

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E-mail address: `masdeu@mat.uab.cat`

Department of Mathematics, University of Warwick, Coventry, United Kingdom

E-mail address: `seveso.marco@gmail.com`

Dipartimento di Matematica "Federigo Enriques", Università di Milano, Via Cesare Saldini 50, 20133 Milano, Italy